Translations of
MATHEMATICAL MONOGRAPHS

Volume 18

Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space

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American Mathematical Society

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INTRODUCTION TO THE THEORY OF LINEAR NONSELFADJOINT OPERATORS

by
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AMERICAN MATHEMATICAL SOCIETY

Providence, Rhode Island 02904

1969

ВВЕДЕНИЕ В ТЕОРИЮ ЛИНЕЙНЫХ НЕСАМОСОПРЯЖЕННЫХ ОПЕРАТОРОВ

И. Ц. ГОХБЕРГ
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Издательство „Наука"
Главная Редакция
Физико-Математической Литературы
Москва 1965

Translated from the Russian by
A. Feinstein

Standard Book Number 821-81568-7
Library of Congress Card Number 67-22348

AUTHORS' PREFACE TO THE ENGLISH EDITION

We were happy to learn that immediately after the publication of our book, the American Mathematical Society had undertaken the publication of an English edition.

We wish to thank the translation editor, Dr. S. H. Gould, for the energy and initiative which he has displayed.

We wish to express our gratitude to Dr. A. Feinstein, who took upon himself the task of translating this book and who has fulfilled it with considerable responsibility in the shortest time. We owe to him, as well as to Professor T. Ando, the correction of a number of inaccuracies, a list of which would, to our surprise, prove to be rather long.

February 23, 1967. I.C. Gohberg, M.G. Kreĭn

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PREFACE

The history of the writing of this book is not altogether usual. However, if all authors shared with their readers the history of the origins of their books, then in all probability we would have to deny this statement.

In the spring of 1959 we decided to revise our paper [1] on the theory of defect numbers, root numbers and indices of linear operators, attempting mainly to include in it various results from the theory of perturbations of selfadjoint operators. In particular, we had decided to complete it with the theory of perturbation determinants.

At this point it was discovered that the method of perturbation determinants also made it possible to obtain new results for nonselfadjoint operators. We decided to discuss these results, in connection with which we began to think about what to do in this area, which was new to us. Thus step by step we changed our original plan.... In roughly a year we sent to the editors of the journal "Uspehi Matematičeskih Nauk" a review paper (in it, however, no space was found for perturbation determinants), in which we attempted to give something of a "bird's eye view" of the current state of investigation in the theory of nonselfadjoint operators. The editors of the journal "voted down" the paper, for which we wish to express our deep and sincere gratitude.

The fact is that the size of the original paper (twelve printer's sheets) exceeded by three times the norm established for review papers, and the editors thought it advisable to recommend the publication of the paper as a separate monograph in the series "Modern problems of mathematics".

Naturally, this suggestion induced us to extend the paper somewhat and, in particular, to restore to its rights the theory of perturbation determinants. We thought that all of this would require not more than two or three months. In actuality the work dragged out for three years. It should be said that even while we were working on the paper we had already established close scientific contact with Soviet mathematicians active in this field. Our colleagues gave us help and information concerning their results (before publication), and thus guarded us against the illusion of completeness. If to this one adds our desire to reflect in this book the investigations into which we ourselves were drawn by this work, one can understand our naïve delusion with respect to dates.

We hope that we are properly understood: we bear no grudge towards those who have given us copies of their papers before publication, who have communicated new, improved versions of proofs and have furnished us with all sorts of oral and written information. On the contrary, we are deeply grateful to all those with whose help we have been able, we hope, to achieve a certain degree of harmony in discussing the wide range of investigations which have come before our eyes, concerned with a large new domain of functional analysis—the theory of linear nonselfadjoint operators. Among all our colleagues who have helped us we wish particularly to mention M. S. Brodskiĭ, S. G. Kreĭn, B. Ja. Levin, V. B. Lidskiĭ, Ju. I. Ljubič, A. S. Markus, V. I. Macaev, and L. A. Sahnovič.

However, we have still not completed our account of how this book came to be written. We hope to continue this account in our monograph on the theory of abstract triangular representations of linear operators and its applications to the theory of canonical differential equations.

The editor of this book, F. V. Širokov, time and again "intruded" with his criticism in depth of the presentation of one topic or another. It often turned out that following such an intrusion we began to better understand even our own researches, and as a result the presentation of them gained in simplicity and clarity.

We wish to express our deep gratitude to F. V. Širokov for his considerable efforts and bold "excesses" of editorial authority.

Odessa, Arcadia
September 1, 1964

I. C. GOHBERG
M. G. KREĬN

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INTRODUCTION

The theory of nonselfadjoint operators in Hilbert space is a young branch of functional analysis. In recent times it has attracted the ever increasing attention of mathematicians and physicists, and sometimes of engineers also. The aim of this book is to present a number of achievements in this field, most of them related to the theory of completely continuous operators.

For a long period the investigations of D. Hilbert and F. Riesz, generalizing Fredholm's theory of integral equations, remained the most essential part of this theory.

In contrast with the theory of selfadjoint operators, no spectral decomposition nor even any theorems on the completeness of the system of root vectors had been obtained until recently in the abstract theory of nonselfadjoint operators.

At the same time, important progress was made in the theory of nonselfadjoint boundary value problems for ordinary differential equations by Birkhoff (in 1908) and J. D. Tamarkin (in 1911). These authors started from the methods of Cauchy and Poincaré, based on the study of the analytic properties of the resolvent of the problem.

But as for boundary value problems for partial differential equations, in view of the difficulty in constructing the resolvent, there was for a long time no similar progress.

This situation changed only in 1951, thanks to progress in the abstract theory of nonselfadjoint operators. In this year the work of M. V. Keldyš appeared, in which he established theorems on the completeness of the eigenvectors and associated vectors and theorems concerning the asymptotic properties of the eigenvalues for a wide class of polynomial bundles of nonselfadjoint operators. These theorems made it possible to obtain important results in boundary value problems for partial differential equations; they also led to strong new results for ordinary differential equations.

At the same time, the asymptotic theorems of M. V. Keldyš for abstract operators made it possible to generalize the original investigations of T. Carleman on the asymptotic behavior of the eigenvalues for boundary value problems for second order elliptic differential operators.

In the same period (1951 1956) M. S. Livšic and a group of his co-workers began the development of a new and profound technique in the theory of nonselfadjoint operators.

The guiding light of these investigations was the idea of generalizing to abstract operators the algebraic result of I. Schur on the reduction of a matrix to triangular form by a unitary transformation. These investigations led M. S. Livšic to the creation of a theory of characteristic matrix-functions and triangular models of a nonselfadjoint operator. The realization of this idea in the abstract theory of operators in infinite-dimensional space entailed many difficulties. An essential role in overcoming these difficulties was played by the deep work of V. P. Potapov in the theory of analytic matrix-functions, which in turn was stimulated by investigations on nonselfadjoint operators.

New progress in the theory of the triangular representation of an operator has taken place in recent years (1958—1962).

Already in 1954 N. Aronszajn and K. Smith established the existence of a nontrivial invariant subspace for every completely continuous operator in a Banach space. Later, in 1958, L. A. Sahnovič first deduced from this geometric fact the existence of triangular models for a wider class of operators, and with the help of these models obtained new analytic results in the theory of nonselfadjoint operators.

At approximately the same period, there appeared in papers of M. S. Brodskiĭ an integral giving an abstract triangular representation of a nonselfadjoint operator. The naturalness and transparency of this representation attracted to it the attention of a new group of mathematicians. In a short time the theory of the new integral and its applications was substantially developed (M. S. Brodskiĭ, I. C. Gohberg and M. G. Kreĭn, V. I. Macaev). This integral led in a natural way to the singling out of remarkable classes of Banach spaces, formed from completely continuous operators and representing ideals of the ring of bounded operators. Certain of these spaces were introduced earlier by J. von Neumann and R. Schatten, and many arose precisely in connection with the theory of the integral giving a triangular representation of operators.

Let us mention, finally, that the "abstract triangular integral" made it possible to solve, for a wide class of operators, the problem of their "factorization" (Gohberg and Kreĭn).

Although geometric and algebraic ideas lie at the foundation of the theory of the abstract triangular integral, the solution of a number of fundamental questions of this theory was achieved only thanks to the

application of various (and sometimes new) tools from the theory of analytic functions.

It should be generally emphasized that the development of the theory of linear operators has drawn more and more widely on the theory of functions and, in particular, analytic functions (the theory of the growth and the distribution of the zeros and poles of entire and meromorphic functions, various generalizations and refinements of the Phragmén-Lindelöf theorem (for example the theorem of Levinson-Sjeberg), the theory of subharmonic functions, Tauberian theorems and others).

The basic channel for the use of the methods of the theory of functions is the natural and by now classical tradition of studying the spectral properties of a linear operator by studying its resolvent as an analytic operator-function. Fairly recently another channel appeared for the use of these methods; it is connected with the perturbation determinants of linear operators (M. G. Kreĭn, B. Ja. Levin, V. I. Macaev).

On the basis of the study of the resolvent, it has been possible in recent years to obtain strong tests for the existence of a "sufficiently complete" set of invariant subspaces of a (not necessarily bounded) operator (J. Wermer, F. Wolf, E. Bishop, V. I. Macaev and Ju. I. Ljubič). This, in turn, made it possible to develop the theory of abstract triangular representations, extending it to a wide class of bounded and even unbounded operators (L. A. Sahnovič, M. S. Brodskiĭ, Gohberg and Kreĭn).

The study of the resolvent has also made it possible to establish theorems concerning the formation of bases from the root vectors of a nonselfadjoint operator and theorems on the summation of expansions in root vectors (M. A. Naĭmark, V. B. Lidskiĭ, A. S. Markus). Moreover, a justification has been obtained for the Fourier method for equations of evolution with a nonselfadjoint operator of one class or another (Lidskiĭ).

Let us mention that the technique for obtaining bounds for the resolvent of a nonselfadjoint operator, developed by T. Carleman, E. Hille and J. D. Tamarkin, M. V. Keldyš, V. B. Lidskiĭ and others, has been substantially improved in recent papers of V. I. Macaev.

In this brief review, as in the book itself, we have not touched upon the interesting direction in the spectral theory of nonselfadjoint operators developed by N. Dunford [1] and a group of his students and followers.

The theory of the abstract triangular representation of operators and the problem of the factorization of abstract operators, and also their applications, will be discussed in another monograph by the authors.

In the present book these problems are left almost untouched, although in various places we allude to isolated facts from the theory of the triangular representation. The fact is that complete results are sometimes attained only by the interaction of three types of methods: methods based on the study of the growth of the resolvent, methods of the theory of perturbation determinants and, lastly, methods of the theory of the triangular representation of an operator.

In recent times appreciable development has taken place in the theory of special operators which become selfadjoint or unitary upon the introduction of a certain indefinite metric in the Hilbert space. The investigations in this theory deserve discussion in a separate monograph; in the present book we have not touched upon them, except for a few cases related to them (Chapter V, §12).

Going on to a brief indication of the contents of the individual chapters, we mention, first of all, that a substantial part of the material is discussed here, with adequate proofs, for the first time. The material as a whole is systematized for the first time in this monograph.

In the first chapter, well-known results of the general theory of bounded nonselfadjoint operators are recalled. As a rule, these results are not specific for Hilbert spaces; they could have been formulated for operators in a Banach space.

In the third chapter we discuss the theory of symmetrically-normed ideals of the ring of bounded operators in a Hilbert space. This chapter includes the basic content of the elegant book of R. Schatten [2], devoted to the von Neumann-Schatten theory of "norm ideals" and "cross-norms". Together with the "veteran ideals" (the ideals of the nuclear operators, Hilbert-Schmidt operators and others), we single out and study new ideals which play important roles in various questions of the theory of nonselfadjoint operators. We are able to develop a treatment of the theory of symmetrically-normed ideals, starting from higher principles, thanks to the systematic use of the theory of the s-numbers of completely continuous operators worked out by H. Weyl, Ky Fan, A. Horn and others. The theory of s-numbers forms the basic content of the second chapter. General theorems on s-numbers of linear operators are also used systematically in the following two chapters.

The fourth chapter is devoted to the theory of perturbation determinants and some of its applications. The use of the extensive apparatus of the modern theory of analytic functions distinguishes this chapter from the others. This apparatus is regarded as auxiliary, and in most cases results of the theory of functions are presented without proof.

In the fifth chapter we discuss various theorems on the completeness of the system of root (eigen- and associated) vectors of a completely continuous operator (operator bundle). In the selection of the material the authors have attempted, on the one hand, to present sufficiently original and strong results and, on the other, to present as fully as possible the various existing methods.

Here we also discuss various results on the growth of the resolvent and theorems on the asymptotic properties of the spectrum (of operators of various classes).

In this chapter, apparently, the fundamental results of M. V. Keldyš are for the first time treated in a sufficiently detailed form.

Considerable attention has also been devoted to theorems on the completeness of the system of root vectors of a dissipative operator.

The last section is devoted to the study of the spectral properties of a selfadjoint quadratic bundle. Here we use the results of almost all the other sections of this chapter. However, as recent investigations have shown (M. G. Kreĭn and H. Langer), for the construction of a complete theory of quadratic selfadjoint bundles it is natural to use various theorems of the theory of operators in spaces with an indefinite metric. In view of this, certain results are presented here without proof.

As a supplement to the theorems on completeness, the authors have thought it appropriate to discuss in Chapter VI the simplest tests for the existence of a basis (of one kind or another) made up of the root vectors of a given linear operator (results of B. R. Mukminov, I. M. Glazman and A. S. Markus). The theory of bases in a Hilbert space is not discussed in texts on functional analysis; for this reason it too is presented in this chapter.

The presentation in this book is carried out in the spirit of the abstract theory of operators: it is illustrated by various applications to the theory of integral equations.

The reader who has some experience in the theory of boundary value problems for differential equations, or an acquaintance with the theory of linear vibrating systems with a finite or infinite number of degrees of freedom, will easily discover how many of the results discussed here find immediate application in each of these fields. A clearer presentation of the possibilities and prospects existing here can be obtained from the interesting survey article by M. V. Keldyš and V. B. Lidskiĭ [1] (cf. also the survey by C. L. Dolph [1] and the report by M. G. Kreĭn and H. Langer [2]).

CHAPTER I

GENERAL THEOREMS ON BOUNDED NONSELFADJOINT OPERATORS

Many of the results in this chapter on nonselfadjoint operators are by no means characteristic of the theory of operators in Hilbert space. These results could be reformulated and proved without essential changes for bounded (and sometimes even for closed unbounded) operators acting in an arbitrary Banach space.

Although the material presented here became available to specialists some time ago, nevertheless this fact has not been reflected in texts either on the theory of operators in Hilbert space, or on the theory of operators in general Banach spaces.

The material of this chapter, presented in the broader context indicated above, can be found in the authors' paper [1].

§1. Notation and some known results

In this section we introduce basic notation and recall some well-known concepts and results from the theory of operators in Hilbert space.

Henceforth $\mathfrak{H}$ denotes a separable Hilbert space.

The domain of a linear operator A, acting in $\mathfrak{H}$, will be denoted by $\mathfrak{D}(A)$, and the set of its values by $\mathfrak{R}(A)$.

Throughout this chapter we shall be concerned with bounded linear operators. For such operators A it will always be assumed that $\mathfrak{D}(A) = \mathfrak{H}$.

We shall denote the set of all bounded linear operators, acting in $\mathfrak{H}$, by $\mathfrak{R} = \mathfrak{R}(\mathfrak{H})$. The *uniform norm* of an operator $A \in \mathfrak{R}$ is the number

$$|A| = \sup_{|\phi|=1} |A\phi|.$$

With this definition of a norm, $\mathfrak{R}$ becomes a complete normed ring.

1. Invariant subspaces of an operator. A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called an *invariant* subspace of the operator A $(\in \mathfrak{R})$ if for any f from $\mathfrak{L}$ the vector Af also belongs to $\mathfrak{L}$.

Let P $(\in \mathfrak{R})$ be an arbitrary projector $(P^2 = P)$.

Obviously the projection subspace $P\mathfrak{H}$ will be invariant with respect to the operator A $(\in \mathfrak{R})$ if and only if

$$PAP = AP. \tag{1.1}$$

If, moreover, the subspace $Q\mathfrak{H}(Q = I - P)$ is invariant for the operator A, then and only then

$$PA = AP. \tag{1.2}$$

If the operator A is selfadjoint and P is an orthoprojector (orthogonal projector: $P = P^*$), then the conditions (1.1) and (1.2) are equivalent.

The following result will be used frequently in the sequel.

1. *If the subspace $\mathfrak{L}$ is an invariant subspace of the bounded linear operator A, then its orthogonal complement $\mathfrak{L}^{\perp} = \mathfrak{H} \ominus \mathfrak{L}$ is an invariant subspace of the operator A^*.*

In fact, let P be the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{L}$, and let $Q = I - P$. Then the condition (1.1) can be written in the form

$$QAP = 0.$$

Going over to the adjoint operators, we obtain

$$PA^*Q = 0,$$

and the assertion follows.

2. *Let $\mathfrak{L}$ be an invariant subspace of the operator $A \in \mathfrak{R}$, and let P be the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{L}$.*

If two of the operators

$$A, \quad PAP + Q, \quad P + QAQ, \tag{1.3}$$

where $Q = I - P$, are invertible, then the third is also, and their inverses are related by

$$(PAP + Q)^{-1} = PA^{-1}P + Q, \tag{1.4}$$

$$(QAQ + P)^{-1} = QA^{-1}Q + P. \tag{1.5}$$

In fact, since $QAP = 0$, the operator $A = (P + Q)A(P + Q)$ is representable in the form

$$PAP + PAQ + QAQ.$$

It follows that

$$A = (QAQ + P)(I + PAQ)(Q + PAP). \tag{1.6}$$

The square of the operator PAQ equals zero. Thus the operator

$I+PAQ$ is invertible and

$$(I+PAQ)^{-1}=I-PAQ.$$

It follows at once from (1.6) that if two of the operators from (1.3) are invertible, then the third is invertible.

If all three of the operators (1.3) are invertible, then

$$A^{-1}=(Q+PAP)^{-1}(I-PAQ)(QAQ+P)^{-1}.$$

Hence we easily conclude that

(1.7) $PA^{-1}P=P(Q+PAP)^{-1}P$ and $QA^{-1}Q=Q(QAQ+P)^{-1}Q.$

Since

$$P(Q+PAP)^{-1}P=(Q+PAP)^{-1}-Q$$

and

$$Q(QAQ+P)^{-1}Q=(QAQ+P)^{-1}-P,$$

(1.4) and (1.5) follow from (1.7).

2. The resolvent and spectrum of an operator. A point λ of the complex plane is called a *regular point* of the operator A $(\in\mathfrak{R})$, if the operator $A-\lambda I$ is invertible, i.e., if there exists a bounded operator $R(\lambda)=R(\lambda,A)$ $=(A-\lambda I)^{-1}$ (called the *resolvent*) such that

$$R(\lambda)(A-\lambda I)=(A-\lambda I)R(\lambda)=I.$$

The set $\rho(A)$ of all regular points of the operator A, called the *resolvent set* of A, is always open.[1] In fact, if $\lambda_0\in\rho(A)$, then it follows from the representation

$$A-\lambda I=A-\lambda_0 I+(\lambda_0-\lambda)I=(A-\lambda_0 I)(I+(\lambda_0-\lambda)R(\lambda_0))$$

that the resolvent exists in the disk

$$|\lambda-\lambda_0|<|R(\lambda_0)|^{-1}$$

and can be obtained from the formula

(1.8) $$R(\lambda)=(I-(\lambda-\lambda_0)R(\lambda_0))^{-1}R(\lambda_0)=\sum_{k=0}^{\infty}(\lambda-\lambda_0)^kR^{k+1}(\lambda_0).$$

At the same time, we see that the resolvent $R(\lambda)$ is a holomorphic

[1] For $A\in\mathfrak{R}$, the point at infinity $\lambda=\infty$ is always added on to $\rho(A)$.

operator-function[2] in the region $\rho(A)$.

For any two points $\lambda, \mu \in \rho(A)$, one immediately verifies the Hilbert identity

$$R(\lambda) - R(\mu) = (\lambda - \mu) R(\lambda) R(\mu).$$

For λ and μ close to each other this identity also follows from (1.8).

The following assertion holds.

3. *Let A be any operator from $\Re$, and let F be any closed set of points of the complex plane.*

If $F \subset \rho(A)$, then one can find $\delta > 0$ such that $F \subset \rho(A + B)$ for all operators $B \in \Re$ satisfying the condition $|B| < \delta$.

This property can be understood as the "continuous dependence" of the resolvent set $\rho(A)$ upon the operator A. Let us prove the assertion: since the operator-function $R(\lambda) = R(\lambda, A)$ is holomorphic in $\rho(A)$ and F is closed, it follows that

$$\max_{\lambda \in F} |R(\lambda)| = \frac{1}{\delta} > 0$$

(even if the set F is unbounded).

For any operator $B \in \Re$ with norm $|B| < \delta$ and any $\lambda \in F$ we have

$$|BR(\lambda)| < 1,$$

and consequently the operator $A + B - \lambda I$ is invertible:

$$(A + B - \lambda I)^{-1} = R(\lambda) \sum_{k=0}^{\infty} (-BR(\lambda))^k.$$

Thus $F \subset \rho(A + B)$.

The complement of the set $\rho(A)$ with respect to the entire complex plane is called the *spectrum* $\sigma(A)$ of the operator A. Thus the spectrum of an operator is always a closed set.

We note that the spectra $\sigma(A)$ and $\sigma(A^*)$ are mirror images of each other with respect to the real axis.

[2] An operator-function $A(\lambda)$ $(\lambda \in G)$ is called a *holomorphic* operator-function in the region G, if $A(\lambda) \in \Re$ for every $\lambda \in G$ and if in a neighborhood of each point $\lambda_0 \in G$ the function $A(\lambda)$ can be expanded in a series

$$A(\lambda) = A(\lambda_0) + \sum_{j=1}^{\infty} (\lambda - \lambda_0)^j C_j, \quad \text{where } C_j \in \Re \ (j = 1, 2, \cdots),$$

which converges in the uniform norm.

The spectrum $\sigma(A)$ contains all *eigenvalues* of the operator A, i.e., all numbers λ for which the equation $(A - \lambda I)\phi = 0$ has at least one nonzero solution $\phi \in \mathfrak{H}$ (an *eigenvector*).

A nonzero vector $\phi \in \mathfrak{H}$ is called a *root vector* of the operator $A \in \mathfrak{R}$, corresponding to the eigenvalue λ_0, if $(A - \lambda_0 I)^n \phi = 0$ for some positive integer n.

The set of all root vectors of the operator A, corresponding to one and the same eigenvalue λ_0, together with the vector $\phi = 0$, forms a lineal[3] $\mathfrak{L}_{\lambda_0}$, which is called the *root lineal*. The dimension $\nu_{\lambda_0} = \nu_{\lambda_0}(A)$ of the lineal $\mathfrak{L}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. If $\nu_{\lambda_0} < \infty$, then the lineal $\mathfrak{L}_{\lambda_0}$ turns out, naturally, to be closed; in this case we shall speak of the *root subspace*.

Obviously, the *eigenspace* $\mathfrak{Z}_{\lambda_0}$ (i.e., the set consisting of the null-vector and all eigenvectors of the operator A corresponding to the value λ_0) is a part of $\mathfrak{L}_{\lambda_0}$ ($\mathfrak{Z}_{\lambda_0} \subseteq \mathfrak{L}_{\lambda_0}$). The dimension $\alpha(\lambda_0)$ of the subspace $\mathfrak{Z}_{\lambda_0}$ is called the *proper multiplicity* of the value λ_0. Thus, the proper multiplicity of any eigenvalue does not exceed its algebraic multiplicity.

3. The integral of F. Riesz. Let Γ be a simple or composite rectifiable contour, which encloses some region G_Γ and lies entirely within the resolvent set $\rho(A)$ ($\Gamma \subset \rho(A)$) of the operator $A \in \mathfrak{R}$. Then $R(\lambda) = (A - \lambda I)^{-1}$ is an analytic operator-function on Γ. Assuming that the contour Γ has positive orientation with respect to the region G_Γ, we form the integral

$$P_\Gamma = -\frac{1}{2\pi i}\int_\Gamma R(\lambda)\, d\lambda.$$

One has the following results of F. Riesz (cf. Riesz and Sz.-Nagy [1]).

4. *The operator P_Γ is a projector which commutes with A, and hence in the decomposition*

$$\mathfrak{H} = \mathfrak{L}_\Gamma + \mathfrak{N}_\Gamma, \quad \textit{where } \mathfrak{L}_\Gamma = P_\Gamma \mathfrak{H},\ \mathfrak{N}_\Gamma = (I - P_\Gamma)\mathfrak{H},$$

both terms $\mathfrak{L}_\Gamma$ and $\mathfrak{N}_\Gamma$ are invariant subspaces of the operator A. Moreover,

a) *the spectrum of the restriction of A to the subspace $\mathfrak{L}_\Gamma$ is contained in the region G_Γ;*

b) *the spectrum of the restriction of A to the subspace $\mathfrak{N}_\Gamma$ lies outside the closure of G_Γ.*

5. *If Γ_1 and Γ_2 are two different contours having the properties indicated*

---

[3] That is, a linear manifold which, generally speaking, is not closed.

above, and the regions G_{Γ_1} *and* G_{Γ_2} *have no points in common, then the projectors corresponding to them are mutually orthogonal, i.e.,*

$$P_{\Gamma_1}P_{\Gamma_2} = P_{\Gamma_2}P_{\Gamma_1} = 0.$$

It follows from the results 4 and 5 that if only a finite number of points $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the spectrum of the operator A lie in the region G_Γ, then

$$P_\Gamma = \sum_{j=1}^{n} P_{\lambda_j}, \quad P_{\lambda_j}P_{\lambda_k} = 0 \qquad (j \neq k), \tag{1.9}$$

where the P_{λ_j} $(j = 1, 2, \cdots, n)$ are projectors which project $\mathfrak{H}$ onto subspaces $P_{\lambda_j}\mathfrak{H}$, invariant with respect to A, in each of which the entire spectrum of A consists of the single point λ_j.

In fact, let γ_j be nonintersecting circles which lie entirely in G_Γ and have centers λ_j; then

$$P_\Gamma = -\sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} R(\lambda)\, d\lambda = \sum_{j=1}^{n} P_{\gamma_j}.$$

Since the projectors P_{γ_j} do not change as the radii of the circles γ_j are decreased, the P_{γ_j} are completely determined by the points λ_j $(P_{\gamma_j} = P_{\lambda_j})$ and have the properties indicated above.

4. Polar representation of a bounded operator. For $A \in \mathfrak{R}$ we denote by $\mathfrak{Z}(A)$ the subspace of zeros of the operator A, i.e., the subspace of all solutions ϕ of the equation $A\phi = 0$. It is well known that

$$\mathfrak{H} = \overline{\mathfrak{R}(A)} \oplus \mathfrak{Z}(A^*) = \overline{\mathfrak{R}(A^*)} \oplus \mathfrak{Z}(A),$$

where the bar denotes the closure of the corresponding lineal.

An operator $U \in \mathfrak{R}$ is said to be *partially isometric* if it maps the subspace $\mathfrak{H} \ominus \mathfrak{Z}(U)$ isometrically onto $\mathfrak{R}(U)$. For a partially isometric operator U the lineal $\mathfrak{R}(U)$ is a closed subspace.

It is easily seen that if U is a partially isometric operator, so is the operator U^*. The operator U defines an isometric mapping of the subspace $\mathfrak{R}(U^*)$ onto $\mathfrak{R}(U)$, and the operator U^* defines the inverse mapping of $\mathfrak{R}(U)$ onto $\mathfrak{R}(U^*)$, so that

$$U^*U = P_1 \quad \text{and} \quad UU^* = P_2,$$

where P_1, P_2 are the orthoprojectors which project $\mathfrak{H}$ onto the subspaces $\mathfrak{R}(U^*)$ and $\mathfrak{R}(U)$, respectively.

Let A be any operator from $\mathfrak{R}$. Then, as is well known, there exists a unique nonnegative operator H such that $H^2 = A^*A$ $(H = (A^*A)^{1/2})$.

For this operator we have

$$|Af|^2 = (Af, Af) = (A^*Af, f) = (Hf, Hf) = |Hf|^2 \qquad (f \in \mathfrak{H}).$$

The operator U, which associates with the vector Hf the vector Af, maps $\mathfrak{R}(H)$ isometrically onto $\mathfrak{R}(A)$. Extending the operator U to all of $\overline{\mathfrak{R}(H)}$ by continuity and setting $U\phi = 0$ for $\phi \in \mathfrak{Z}(H)$, we obtain a partially isometric operator.

It is easily seen that

$$\overline{\mathfrak{R}(H)} = \overline{\mathfrak{R}(H^2)} = \overline{\mathfrak{R}(A^*A)} = \overline{\mathfrak{R}(A^*)}.$$

Thus every bounded linear operator A admits a representation in the form

$$A = UH, \tag{1.10}$$

where $H = (A^*A)^{1/2}$, and U is a partially isometric operator which maps the subspace $\overline{\mathfrak{R}(A^*)}$ isometrically onto $\overline{\mathfrak{R}(A)}$.

The representation (1.10) is called the *polar representation* of the operator A.

One can easily prove the following properties of the operators which appear in the polar representation of the operator A:

1) $U^*A = H$;
2) $H_1 = UHU^*$, $H = U^*H_1U$, where $H_1 = (AA^*)^{1/2}$;
3) $A = H_1U$, $H_1 = AU^*$.

5. The dimension of an operator. Finite-dimensional operators. By the *dimension of the operator* A is meant the number $r(A)$ ($\leq \infty$), equal to the dimension of the subspace $\overline{\mathfrak{R}(A)}$.

It is easily seen that

$$r(A) = r((A^*A)^{1/2}) = r((AA^*)^{1/2}) = r(A^*).$$

An operator $A \in \mathfrak{R}$ is said to be *finite-dimensional* if $r(A) < \infty$. Obviously, every operator K defined by

$$Kf = \sum_{j=1}^{n} (f, \phi_j)\psi_j \qquad (f \in \mathfrak{H}), \tag{1.11}$$

where $\{\phi_j\}_1^n$ and $\{\psi_j\}_1^n$ are arbitrary systems of vectors from $\mathfrak{H}$, is a finite-dimensional operator, and $r(K) \leq n$. It is easy to show that

$$r(K) = \min(\dim \mathfrak{L}_1, \dim \mathfrak{L}_2),$$

where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are the linear hulls of the systems $\{\phi_j\}_1^n$ and $\{\psi_j\}_1^n$, respectively.

Every finite-dimensional operator K can be represented in the form (1.11) with $n = r(K)$. In fact, let $\{\psi_j\}_1^n$ (where $n = r(K)$) be any basis of the subspace $\mathfrak{R}(K)$. Then for every vector $f \in \mathfrak{H}$,

$$Kf = \sum_{j=1}^{n} c_j \psi_j, \tag{1.12}$$

where the c_j, $j = 1, \cdots, n$, are complex numbers depending upon f. Let $\{\chi_j\}_1^n$ be any system of vectors which is biorthogonal to the system $\{\psi_j\}_1^n$, i.e.,

$$(\psi_j, \chi_k) = \delta_{jk} \qquad (j, k = 1, 2, \cdots, n).$$

Then

$$c_j = (Kf, \chi_j),$$

or

$$c_j = (f, \phi_j) \qquad (j = 1, 2, \cdots, n), \tag{1.13}$$

where $\phi_j = K^* \chi_j$. Substituting (1.13) into (1.12), we obtain (1.11).

Henceforth, the operator K, defined by (1.11), will be written in the form

$$K = \sum_{j=1}^{n} (\cdot, \phi_j) \psi_j.$$

§2. Normal points of a bounded operator

1. We shall call an eigenvalue λ_0 of an operator A $(\in \mathfrak{R})$ a *normal eigenvalue*[4] of A if

1) the algebraic multiplicity of λ_0 is finite, and

2) the space $\mathfrak{H}$ breaks up into the direct sum of subspaces

$$\mathfrak{H} = \mathfrak{L}_{\lambda_0} \dotplus \mathfrak{N}_{\lambda_0}, \tag{2.1}$$

where $\mathfrak{L}_{\lambda_0}$ is the root subspace of the operator A corresponding to λ_0, and $\mathfrak{N}_{\lambda_0}$ is an invariant subspace of A in which the operator $A - \lambda_0 I$ is invertible.

The decomposition (2.1) of the space $\mathfrak{H}$ is unique. Indeed, if ν is the algebraic multiplicity of λ_0 (i.e., $\nu = \dim \mathfrak{L}_{\lambda_0}$), then

$$(A - \lambda_0 I)^\nu \mathfrak{L}_{\lambda_0} = 0.$$

Since the operator $A - \lambda_0 I$ is invertible in $\mathfrak{N}_{\lambda_0}$, we obtain

[4] In the authors' paper [1], instead of the term "normal eigenvalue" the expression "number corresponding to a normally splitting finite-dimensional root subspace" was used.

$$(A - \lambda_0 I)^\nu \mathfrak{H} = (A - \lambda_0 I)^\nu \mathfrak{L}_{\lambda_0} + (A - \lambda_0 I)^\nu \mathfrak{N}_{\lambda_0} = \mathfrak{N}_{\lambda_0}.$$

Thus $\mathfrak{N}_{\lambda_0} = (A - \lambda_0 I)^\nu \mathfrak{H}$.

THEOREM 2.1. *In order that a point λ_0 of the spectrum of an operator $A \in \mathfrak{R}$ be a normal eigenvalue, it is necessary and sufficient that it be an isolated point of the spectrum of A and that the projector corresponding to it*

$$P_{\lambda_0} = -\frac{1}{2\pi i}\int_{|\lambda - \lambda_0| = \delta} R(\lambda)\, d\lambda \tag{2.2}$$

be finite-dimensional.

If λ_0 is a normal eigenvalue, then P_{λ_0} projects $\mathfrak{H}$ onto the root subspace $\mathfrak{L}_{\lambda_0}$.

PROOF. Let us assume that λ_0 is a normal eigenvalue of the operator A. We denote by A_1 and A_2 the restrictions of A to the subspaces $\mathfrak{L}_{\lambda_0}$ and $\mathfrak{N}_{\lambda_0}$, respectively. As was just mentioned, the operator $A_1 - \lambda_0 I$ is nilpotent: $(A_1 - \lambda_0 I)^\nu = 0$.

Let us denote by n the smallest positive integer such that $(A_1 - \lambda_0 I)^n = 0$. Then, setting $B_1 = A_1 - \lambda_0 I$, we have

$$\begin{aligned} -(\lambda - \lambda_0)^n I &= B_1^n - (\lambda - \lambda_0)^n I \\ &= (A_1 - \lambda I)\left[(\lambda - \lambda_0)^{n-1} I + (\lambda - \lambda_0)^{n-2} B_1 + \cdots + B_1^{n-1}\right]. \end{aligned}$$

Hence

$$-(A_1 - \lambda I)^{-1} = (\lambda - \lambda_0)^{-1} I + \sum_{j=1}^{n-1} (\lambda - \lambda_0)^{-j-1} B_1^j.$$

On the other hand, the operator $A_2 - \lambda_0 I$ is invertible in the subspace $\mathfrak{N}_{\lambda_0}$. Hence for all λ from the disk $|\lambda - \lambda_0| < 1/|(A_2 - \lambda_0 I)^{-1}|$ the resolvent

$$(A_2 - \lambda I)^{-1} = R_0 + (\lambda - \lambda_0) R_0^2 + \cdots + (\lambda - \lambda_0)^n R_0^{n+1} + \cdots$$

exists, where $R_0 = (A_2 - \lambda_0 I)^{-1}$. It follows that all points λ satisfying the inequality $0 < |\lambda - \lambda_0| < |R_0|^{-1}$ are regular points of A, and for these points the resolvent $R(\lambda)$ is given by the formula

$$\begin{aligned} R(\lambda) &= (A - \lambda I)^{-1} \\ &= -((\lambda - \lambda_0)^{-n} B_1^{n-1} + \cdots + (\lambda - \lambda_0)^{-2} B_1 + (\lambda - \lambda_0)^{-1} I)P \\ &\quad + \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_0^{k+1} (I - P), \end{aligned} \tag{2.3}$$

where P is the projector which projects the space $\mathfrak{H}$ onto the subspace $\mathfrak{L}_{\lambda_0}$ along $\mathfrak{N}_{\lambda_0}$ ($P\mathfrak{H} = \mathfrak{L}_{\lambda_0}, P\mathfrak{N}_{\lambda_0} = 0$).

Integrating both sides of (2.3) over a sufficiently small circle with

center at the point λ_0, we find that

$$P = P_{\lambda_0} = -\frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\delta} R(\lambda)\,d\lambda.$$

Thus the necessity of the hypotheses of the theorem is proved. Let us prove their sufficiency.

Let λ_0 be an isolated point of the spectrum of A, and let the projector (2.2) be finite-dimensional. Then the space $\mathfrak{H}$ can be broken up into the direct sum of the subspaces $\mathfrak{M}_{\lambda_0} = P_{\lambda_0}\mathfrak{H}$ and $\mathfrak{N}_{\lambda_0} = (I - P_{\lambda_0})\mathfrak{H}$, which are invariant with respect to A. The restriction of A to the subspace $\mathfrak{M}_{\lambda_0}$ has a unique spectral point $\lambda = \lambda_0$. Hence

(2.4) $$(A - \lambda_0 I)^\nu \mathfrak{M}_{\lambda_0} = 0 \quad (\nu = \dim \mathfrak{M}_{\lambda_0}).$$

The point λ_0 is a regular point of the restriction of A to the subspace $\mathfrak{N}_{\lambda_0}$. It follows from (2.4) that $\mathfrak{M}_{\lambda_0}$ is the root subspace of the operator A corresponding to the number λ_0. Since $\mathfrak{M}_{\lambda_0} \dotplus \mathfrak{N}_{\lambda_0} = \mathfrak{H}$, it follows that λ_0 is a normal eigenvalue. The theorem is proved.

For completeness we shall state without proof another result concerning normal eigenvalues of an operator, which will not be used further on (cf. Gohberg and Kreĭn [1]).

In order that λ_0 be a normal eigenvalue of an operator $A \in \mathfrak{R}$, it is necessary and sufficient that: a) *λ_0 be an isolated point of the spectrum of A,* b) *the algebraic multiplicity of λ_0 be finite, and* c) *the range $(A - \lambda_0 I)\mathfrak{H}$ of the operator $A - \lambda_0 I$ be closed.*

2. We note three results which follow from Theorem 2.1.

1. *If λ_0 is a normal eigenvalue of the operator A of arbitrary algebraic multiplicity, then $\bar{\lambda}_0$ will be a normal eigenvalue of the operator A^* of the same algebraic multiplicity.*

In fact, the spectra $\sigma(A)$ and $\sigma(A^*)$ are mirror images of each other with respect to the real axis, and thus the point $\bar{\lambda}_0$ is an isolated point of $\sigma(A^*)$.

On the other hand, if λ_0 and $\bar{\lambda}_0$ are isolated points of the respective spectra $\sigma(A)$ and $\sigma(A^*)$, then always

(2.5) $$P_{\bar{\lambda}_0}(A^*) = [P_{\lambda_0}(A)]^*.$$

2. *Let λ_0 be a normal eigenvalue of the operator A $(\in \mathfrak{R})$ and let the decomposition corresponding to it be*

$$\mathfrak{H} = \mathfrak{L}_{\lambda_0} \dotplus \mathfrak{N}_{\lambda_0};$$

then to the normal eigenvalue $\bar{\lambda}_0$ of the operator A^* there corresponds the

decomposition

$$\mathfrak{H} = \mathfrak{L}_{\lambda_0}^* \dotplus \mathfrak{N}_{\bar{\lambda}_0}^*,$$

where

$$\mathfrak{L}_{\bar{\lambda}_0}^* = \mathfrak{N}_{\lambda_0}^{\perp} \quad \textit{and} \quad \mathfrak{N}_{\bar{\lambda}_0}^* = \mathfrak{L}_{\lambda_0}^{\perp}.$$

This result is a simple consequence of (2.5).

3. *If λ_0 is a normal eigenvalue of the operator A, then the equations*

$$(A - \lambda_0 I)\,\phi = 0 \quad \textit{and} \quad (A^* - \bar{\lambda}_0 I)\,\psi = 0$$

have the same number of linearly independent solutions.

In fact, the number of linearly independent solutions of the equation $(A^* - \bar{\lambda}_0 I)\,\psi = 0$ equals the dimension of the subspace $\mathfrak{H} \ominus \mathfrak{R}(A - \lambda_0 I)$. Since the restriction of the operator $A - \lambda_0 I$ to the subspace $\mathfrak{N}_{\lambda_0} = (I - P_{\lambda_0})\,\mathfrak{H}$ is invertible, we have

$$\begin{aligned}\dim(\mathfrak{H} \ominus \mathfrak{R}(A - \lambda_0 I)) &= \dim(\mathfrak{L}_{\lambda_0} \ominus (A - \lambda_0 I)\,\mathfrak{L}_{\lambda_0}) \\ &= \dim(\mathfrak{L}_{\lambda_0} \ominus \mathfrak{R}(A_1 - \lambda_0 I_1)),\end{aligned}$$

where $\mathfrak{L}_{\lambda_0} = P_{\lambda_0}\mathfrak{H}$ and A_1 and I_1 are the restrictions of the operators A and I to $\mathfrak{L}_{\lambda_0}$.

The subspace $\mathfrak{L}_{\lambda_0}$ is finite-dimensional and consequently the dimension of the subspace of solutions of the equation $(A_1 - \lambda_0 I_1)\,\phi = 0$ coincides with the dimension of the subspace $\mathfrak{L}_{\lambda_0} \ominus \mathfrak{R}(A_1 - \lambda_0 I_1)$. Finally, the equations

$$(A_1 - \lambda_0 I_1)\,\phi = 0 \quad \text{and} \quad (A - \lambda_0 I)\,\phi = 0$$

have the same solutions. The result is proved.

3. We shall call a point λ_0 of the complex plane a *normal point* of the operator A, if it is either a regular point of A or a normal eigenvalue of A. Since $\rho(A)$ is an open set and every normal eigenvalue of the operator A is an isolated point of the spectrum of A, the set $\tilde{\rho}(A)$ $(\supset \rho(A))$ of all normal points of A is always open.

THEOREM 2.2. *Let Γ be a rectifiable contour which encloses some region G_Γ and consists of regular points of the operator A $(\in \mathfrak{R})$. The region G_Γ consists of normal points of A if and only if the projector*

$$P_\Gamma = -\frac{1}{2\pi i}\int_\Gamma (A - \lambda I)^{-1} d\lambda$$

is finite-dimensional.

When this condition is fulfilled, either G_Γ does not contain points of the spectrum of A, or the spectrum of A in G_Γ consists of a finite number of normal eigenvalues. In the latter case the subspace $P_\Gamma \mathfrak{H}$ will be the direct sum of all the root subspaces of the operator A corresponding to the eigenvalues from G_Γ.

PROOF. Suppose that the portion of the spectrum of the operator A which is contained in G_Γ consists of a finite number of normal points $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, by virtue of the relation (1.9),

$$P_\Gamma = P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_n} \qquad (P_{\lambda_j} P_{\lambda_k} = 0 \text{ for } j \neq k), \tag{2.6}$$

where the projector P_{λ_j} $(j = 1, 2, \cdots, n)$ projects the space $\mathfrak{H}$ onto the finite-dimensional root subspace $\mathfrak{L}_{\lambda_j}$ of the operator A corresponding to the eigenvalue λ_j. Hence the projector P_Γ is finite-dimensional and

$$\mathfrak{L}_\Gamma = P_\Gamma \mathfrak{H} = \sum_{j=1}^{n} P_{\lambda_j} \mathfrak{H} = \mathfrak{L}_{\lambda_1} \dotplus \mathfrak{L}_{\lambda_2} \dotplus \cdots \dotplus \mathfrak{L}_{\lambda_n}.$$

Conversely, suppose that the projector P_Γ is finite-dimensional.

Then $\mathfrak{H}$ can be broken up into the direct sum of the subspaces $\mathfrak{L}_\Gamma$ and $\mathfrak{N}_\Gamma = (I - P_\Gamma)\mathfrak{H}$, which are invariant with respect to the operator A.

Let us denote by A_1 and A_2 the restrictions of A to the subspaces $\mathfrak{L}_\Gamma$ and $\mathfrak{N}_\Gamma$, respectively. The subspace $\mathfrak{L}_\Gamma$ is finite-dimensional; thus the spectrum of the operator A_1 consists of a finite number of eigenvalues λ_j $(j = 1, 2, \cdots, n;\ \lambda_j \in G_\Gamma)$. By a well-known result in the theory of finite matrices, the subspace $\mathfrak{L}_\Gamma$ can be broken up into the direct sum of subspaces $\mathfrak{L}_j$, invariant with respect to A_1 and such that the operator $A_1 - \lambda_j I$ is nilpotent in the corresponding subspace $\mathfrak{L}_j$. Obviously, $A_1 - \lambda_j I$ will be invertible on all subspaces $\mathfrak{L}_k$ with $k \neq j$.

The operator $A_2 - \lambda I$ is invertible for all $\lambda \in G_\Gamma$; hence the portion of the spectrum of A which lies in the region G_Γ coincides with the spectrum of A_1. Thus A has a finite number of eigenvalues λ_j $(j = 1, 2, \cdots, n)$ inside G_Γ, to which there correspond the finite-dimensional root subspaces $\mathfrak{L}_j$. The points λ_j are normal points, since the space $\mathfrak{H}$ can be broken up into the direct sum of subspaces which are invariant with respect to A:

$$\mathfrak{H} = \mathfrak{L}_j \dotplus \mathfrak{N}_j \quad (j = 1, 2, \cdots, n),$$

and the operator $A - \lambda_j I$ is invertible in the subspace

$$\mathfrak{N}_j = \mathfrak{N}_\Gamma \dotplus \sum_{k \neq j} \mathfrak{L}_k \quad (j = 1, 2, \cdots, n).$$

The theorem is proved.

§3. Stability of the root multiplicities of an operator

Let $A \in \Re$ and let Γ be a rectifiable contour which encloses a region G_Γ and which has the following properties:

a) that part of the spectrum of the operator A which lies in G_Γ consists of a finite number of normal points $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A;

b) the contour Γ consists of regular points of A.

By the *root multiplicity*[5] of A corresponding to the contour Γ we shall mean the sum of the algebraic multiplicities of all the eigenvalues λ_j $(j = 1, 2, \cdots, n)$ of A which lie in the region G_Γ, i.e., the number

$$\nu_\Gamma(A) = \sum_{j=1}^{n} \nu_{\lambda_j}(A).$$

According to (2.6),

$$\nu_\Gamma(A) = \dim P_\Gamma \mathfrak{H},$$

where

$$P_\Gamma = -\frac{1}{2\pi i} \int_\Gamma (A - \lambda I)^{-1} d\lambda.$$

We shall precede Theorem 3.1, which concerns the stability of the root multiplicities of an operator under perturbation by a small term, by a lemma.

LEMMA 3.1 (B. SZ.-NAGY [1, 2]). *Let P and Q be projectors. If $|P - Q| < 1$, then the subspaces $P\mathfrak{H}$ and $Q\mathfrak{H}$ have the same dimension:*

$$\dim P\mathfrak{H} = \dim Q\mathfrak{H}. \tag{3.1}$$

PROOF. If $|P - Q| < 1$, then the operator $I - (P - Q)$ is invertible:

$$(I - P + Q)^{-1} = I + \sum_{j=1}^{\infty} (P - Q)^j,$$

and thus $(I - P + Q)\mathfrak{H} = \mathfrak{H}$. Hence

$$P(I - P + Q)\mathfrak{H} = P\mathfrak{H},$$

and since $P^2 = P$, we have

$$PQ\mathfrak{H} = P\mathfrak{H} \quad \text{or} \quad P\Re(Q) = \Re(P).$$

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[5] In the authors' paper [1] instead of the term "root multiplicity" the term "root number" was used.

Thus P maps the subspace $Q\mathfrak{H}$ onto all of the subspace $P\mathfrak{H}$. Moreover, for any $f \in Q\mathfrak{H}$

$$|Pf| = |f + (P - Q)f| \geqq (1 - |P - Q|)|f|,$$

from which it follows that P maps $Q\mathfrak{H}$ onto $P\mathfrak{H}$ in a one-to-one and bicontinuous manner. This implies (3.1). The lemma is proved.

THEOREM 3.1. *Let Γ be a rectifiable closed curve which encloses some region G_Γ and has the properties* a) *and* b) *with respect to the operator $A \in \mathfrak{R}$. Then there exists $\rho > 0$ such that for all operators $B \in \mathfrak{R}$ satisfying the condition $|B - A| < \rho$, the contour Γ also has the properties* a), b) *with respect to the operator B, and $\nu_\Gamma(A) = \nu_\Gamma(B)$.*

PROOF. Denoting as before the resolvent of the operator A by $R(\lambda)$, we put

$$\delta = \frac{1}{\max\limits_{\lambda \in \Gamma} |R(\lambda)|}.$$

Then

$$\rho = \delta^2 / \left(\delta + \frac{l}{2\pi}\right) \quad (< \delta),$$

where l is the length of the contour Γ, will be the number whose existence is asserted in the theorem.

Indeed, let B be any operator from $\mathfrak{R}$ satisfying the inequality

$$|A - B| < \rho. \tag{3.2}$$

All the points $\lambda \in \Gamma$ are regular points of B, since for $\lambda \in \Gamma$ there exists

$$(B - \lambda I)^{-1} = R(\lambda)\left(I + \sum_{j=1}^{\infty} [(A - B)R(\lambda)]^j\right). \tag{3.3}$$

The convergence of the series on the right is guaranteed by the inequality

$$|(B - A)R(\lambda)| \leqq |B - A|\,|R(\lambda)| < 1 \qquad (\lambda \in \Gamma),$$

which follows from (3.2).

We now introduce the projector

$$\widetilde{P}_\Gamma = -\frac{1}{2\pi i}\int_\Gamma (B - \lambda I)^{-1} d\lambda.$$

It follows from (3.2) and (3.3) that

$$|\widetilde{P}_\Gamma - P_\Gamma| = \frac{1}{2\pi}\left|\int_\Gamma R(\lambda)\sum_{j=1}^{\infty}[(A-B)R(\lambda)]^j d\lambda\right|$$
$$\leq \frac{l}{2\pi}\max_{\lambda\in\Gamma}\frac{|A-B|\,|R(\lambda)|^2}{1-|A-B|\,|R(\lambda)|}.$$

Using the inequalities

$$|A-B| < \rho = 2\pi\delta^2(l+2\pi\delta)^{-1},\ |R(\lambda)| < \delta^{-1} \qquad (\lambda\in\Gamma),$$

we obtain

$$|\widetilde{P}_\Gamma - P_\Gamma| < 1.$$

Hence, according to Lemma 3.1,

$$\dim \widetilde{P}_\Gamma \mathfrak{H} = \dim P_\Gamma \mathfrak{H}, \tag{3.4}$$

and thus the projector $\widetilde{P}_\Gamma$ is finite-dimensional along with P_Γ.

But then, by Theorem 2.2, the contour Γ will have the properties a), b) with respect to the operator B also. Moreover, the equality (3.4) shows that

$$\nu_\Gamma(A) = \nu_\Gamma(B).$$

§ 4. Some spectral properties of completely continuous operators

We shall denote the set of all linear completely continuous operators, acting in $\mathfrak{H}$, by $\mathfrak{S}_\infty$. As is well-known, $\mathfrak{S}_\infty$ is a *two-sided ideal* in the ring $\mathfrak{R}$, and moreover is *selfadjoint* (or *symmetric*) and *closed*. This means that it has the following properties:

1) $\mathfrak{S}_\infty$ is a linear set, and if $X\in\mathfrak{R}$ and $Y\in\mathfrak{S}_\infty$, then $XY\in\mathfrak{S}_\infty$ and $YX\in\mathfrak{S}_\infty$.

2) If $X\in\mathfrak{S}_\infty$, then $X^*\in\mathfrak{S}_\infty$.

3) $\mathfrak{S}_\infty$ is closed in $\mathfrak{R}$ ($\mathfrak{S}_\infty = \overline{\mathfrak{S}}_\infty$).

For reasons which will be clarified later, the uniform norm $|X|$ for $X\in\mathfrak{S}_\infty$ will sometimes be denoted by $|X|_\infty$.

1. Root subspaces of a completely continuous operator. By a well-known theorem of Hilbert (cf. Riesz and Sz.-Nagy [**1**]), every nonzero point of the spectrum of a completely continuous operator A is a normal point of A. Hence such an operator has at most a countable number of spectral points, which can have as a limit point only the point $\lambda = 0$. The latter always (for $\dim\mathfrak{H} = \infty$) belongs to the spectrum of A ($\in\mathfrak{S}_\infty$).

We shall denote the sum of the algebraic multiplicities of all the nonzero eigenvalues of the operator A ($\in\mathfrak{S}_\infty$) by $\nu(A)$ ($\leq\infty$).

The dimension $r(A)$ of an operator A ($\in \mathfrak{S}_\infty$) is related to $\nu(A)$ by the inequality

$$\nu(A) \leqq r(A). \tag{4.1}$$

If $r(A) = \infty$, the inequality (4.1) is trivial. In the case where $r(A) < \infty$, the proof of this inequality reduces to the proof for an operator acting in a finite-dimensional space.

If the operator A is completely continuous, then (by assertion 1 of §2) we have $\nu(A) = \nu(A^*)$.

Henceforth we shall denote by $\{\lambda_j(A)\}_1^{\nu(A)}$ the sequence consisting of all the nonzero eigenvalues of A, arbitrarily enumerated in order of decreasing modulus, and where in this enumeration every eigenvalue is counted as many times as its algebraic multiplicity.[6]

An operator A is called a *Volterra operator* if it is completely continuous and has no nonzero eigenvalues ($\nu(A) = 0$).

From the equality $\nu(A) = \nu(A^*)$ for completely continuous operators it follows that if A is a Volterra operator so is A^*.

Let us denote by $\mathfrak{E}_A$ the closed linear hull of all the root subspaces $\mathfrak{L}_{\lambda_j}$ ($\lambda_j = \lambda_j(A)$) of the completely continuous operator A. Obviously, the subspace $\mathfrak{E}_A$ is invariant with respect to A, and $\nu(A) = \dim \mathfrak{E}_A$.

LEMMA 4.1 (SCHUR'S LEMMA). *Let $\hat{A}$ be the operator induced in $\mathfrak{E}_A$ by the completely continuous operator A. Then there exists an orthonormal basis $\{\omega_j\}_1^{\nu(A)}$ of $\mathfrak{E}_A$ for which the matrix of the operator $\hat{A}$ has triangular form, so that*

$$A\omega_j = a_{j1}\omega_1 + a_{j2}\omega_2 + \cdots + a_{jj}\omega_j \qquad (j = 1, 2, \cdots, \nu(A)), \tag{4.2}$$

where

$$a_{jj} = (A\omega_j, \omega_j) = \lambda_j(A) \qquad (j = 1, 2, \cdots, \nu(A)). \tag{4.3}$$

PROOF. If we choose a Jordan basis in every root subspace $\mathfrak{L}_j$ ($= \mathfrak{L}_{\lambda_j(A)}$) ($j = 1, 2, \cdots, \nu(A)$) and enumerate the vectors of these bases in succession, we obtain a sequence $\{\phi_j\}_1^{\nu(A)}$ for each vector of which one of the two equalities $A\phi_j = \lambda_j\phi_j$, $A\phi_j = \lambda_j\phi_j + \phi_{j-1}$ ($\lambda_j = \lambda_j(A)$) is fulfilled.

It is easily seen that the orthonormal basis $\{\omega_j\}_1^{\nu(A)}$ of the subspace $\mathfrak{E}_A$ (the *Schur system* of the operator A), obtained from the system $\{\phi_j\}$ by successive orthogonalization, will have the properties (4.2) and (4.3).

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[6] In general, the notation $j = 1, 2, \cdots, \nu(A)$ will denote the enumeration over all eigenvalues, numbered in the order of decreasing modulus. The same remark holds concerning the sum $\sum_{j=1}^{\nu(A)}$ and the product $\prod_{j=1}^{\nu(A)}$.

REMARK 4.1. Lemma 4.1 can be given a substantially more general form. *Let $A \in \mathfrak{R}$, let $\{\lambda_j(A)\}$ be some set of its normal eigenvalues, and let $\mathfrak{E}$ be the closed linear hull of all the root vectors corresponding to these eigenvalues.*

Then one can always choose an orthonormal basis $\{\omega_j\}$ of $\mathfrak{E}$, for which the relations (4.2) *and* (4.3) *are fulfilled.*

LEMMA 4.2. *Let A be a completely continuous operator for which $\mathfrak{E}_A \neq \mathfrak{H}$, and let Q_A be the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{E}_A^{\perp}$. Then $Q_A A Q_A$ is a Volterra operator.*

PROOF. Let us denote by

$$\mathfrak{N}_j^* \ (= \mathfrak{N}_{\bar{\lambda}_j}(A^*)) \quad (j = 1, 2, \cdots, \nu(A))$$

the subspaces complementary to the root lineals $\mathfrak{L}_{\bar{\lambda}_j}(A^*)$ of the operator A^*, and by $\mathfrak{M}$ the intersection of all the subspaces $\mathfrak{N}_j^*$. By assertion 2 of §2, a vector f $(\in \mathfrak{H})$ belongs to the subspace $\mathfrak{M}$ if and only if it is orthogonal to all of the lineals $\mathfrak{L}_j$ $(j = 1, 2, \cdots, \nu(A))$. Consequently, the subspaces $\mathfrak{M}$ and $\mathfrak{E}_A$ are orthogonal and

$$\mathfrak{M} \oplus \mathfrak{E}_A = \mathfrak{H}.$$

It follows from assertion 1 of §1 that the subspace $\mathfrak{M}$ is invariant with respect to A^* and that the operator $\hat{A}^*$, induced in $\mathfrak{M}$ by the operator A^*, is a Volterra operator. Therefore the operator $Q_A A^* Q_A$ and its adjoint $Q_A A Q_A$ will be Volterra operators.

REMARK 4.2. As with Lemma 4.1, Lemma 4.2 can be given a more general form.

Let $A \in \mathfrak{R}$, let $\mathscr{S} = \{\lambda_j(A)\}$ be some set of normal eigenvalues of A, and let Q be the orthoprojector which projects $\mathfrak{H}$ onto the orthogonal complement of the linear hull of all the root vectors of A, corresponding to these eigenvalues.

Then the spectrum of the operator QAQ consists of zero, all points $\lambda \in (\sigma(A) \setminus \overline{\mathscr{S}})$ and, possibly, some points $\lambda \in (\overline{\mathscr{S}} \setminus \mathscr{S})$ (i.e., limit points of $\mathscr{S}$).

In particular, if the set $\mathscr{S} = \{\lambda_j(A)\}$ consists of the entire nonzero spectrum of the operator A, then the spectrum of QAQ consists only of zero.

2. The limit of a sequence of completely continuous operators. We start with the following simple result.

THEOREM 4.1. *The limit in the uniform norm of a sequence of Volterra operators is a Volterra operator.*

PROOF. Let A_n $(n = 1, 2, \cdots)$ be a sequence of Volterra operators and let A be its limit in the uniform norm (since $\mathfrak{S}_\infty$ is closed, the operator A automatically belongs to $\mathfrak{S}_\infty$). Let us assume that the operator A has at least one eigenvalue $\lambda_0 \neq 0$. Then by Theorem 3.1 every operator A_n for n sufficiently large has at least one eigenvalue in the disk $|\lambda - \lambda_0| < |\lambda_0|/2$. This contradicts the fact that the A_n $(n = 1, 2, \cdots)$ are Volterra operators. The theorem is proved.

The following more general theorem is somewhat more difficult to prove.

THEOREM 4.2. *Suppose that a sequence of operators A_n $(n = 1, 2, \cdots)$ from $\mathfrak{S}_\infty$ tends in the uniform norm to an operator A $(\in \mathfrak{S}_\infty)$.*

If $\nu(A) = \infty$, then

$$\lim_{n \to \infty} \nu(A_n) = \infty$$

and for an appropriate enumeration of the eigenvalues of the operators A_n we will have

$$\lim_{n \to \infty} \lambda_j(A_n) = \lambda_j(A) \quad (j = 1, 2, \cdots).$$

If $\nu(A) < \infty$, then for every sufficiently small $\epsilon > 0$ there exists a positive integer n_ϵ such that for $n > n_\epsilon$ there exist outside the disk $|\lambda| \leqq \epsilon$ exactly $\nu(A)$ eigenvalues of the operator A_n, for which, by an appropriate enumeration, we will have

$$\lim_{n \to \infty} \lambda_j(A_n) = \lambda_j(A) \quad (j = 1, 2, \cdots, \nu(A)).$$

PROOF. Let $\{\gamma_j\}$ be a sequence of nonintersecting disks with centers at all the distinct eigenvalues of the operator A. If $\nu(A) < \infty$, then we also construct a disk γ_0 with center at the point $\lambda = 0$ and of sufficiently small radius.

By Theorem 3.1 there exists a positive integer N_1 such that for all $n > N_1$ there lie in the disk γ_1 exactly as many eigenvalues of the operator A_n (taking their algebraic multiplicities into account) as the (algebraic) multiplicity κ_1 of the first eigenvalue of the operator A, which is the center of the disk γ_1. We label these eigenvalues of the operator A_n with the integers $1, \cdots, \kappa_1$.

By the same theorem,

$$\lim_{n \to \infty} \lambda_j(A_n) = \lambda_j(A) \quad (j = 1, 2, \cdots, \kappa_1).$$

Let us continue this process. We find a number $N_2 \geqq N_1$ such that

for all $n > N_2$ there lie in the disk γ_2 exactly κ_2 eigenvalues of the operator A_n, where κ_2 is the multiplicity of that eigenvalue of the operator A which is the center of the disk γ_2. To these eigenvalues of the operator A_n we assign the integers from $\kappa_1 + 1$ to $\kappa_1 + \kappa_2$. Again, we have

$$\lim_{n\to\infty} \lambda_j(A_n) = \lambda_j(A) \quad (j = \kappa_1 + 1, \cdots, \kappa_1 + \kappa_2).$$

To complete the proof of the theorem for the case $\nu(A) = \infty$ it remains to continue this process indefinitely and to assign, to those eigenvalues of any operator A_n which remain unlabelled, any "free" numbers.

For the case $\nu(A) < \infty$, in addition to what has been said, it follows from Theorem 3.1 and the result 3 of §1 that for sufficiently large n all the eigenvalues of the operator A_n, except for the first $\nu(A)$, lie in the disk γ_0.

The theorem is proved.

§ 5. A theorem on holomorphic operator-functions and its corollaries.

We shall begin with a result which clarifies the structure of the spectrum of a holomorphic operator-function, all of whose values are completely continuous operators.

LEMMA 5.1. *Let $A(\lambda)$ be an operator-function, holomorphic in a neighborhood U_{λ_0} of the point λ_0, and suppose that all the values of $A(\lambda)$ are completely continuous operators.*

Then there exists $\epsilon > 0$ such that for all values λ from the punctured disk

$$0 < |\lambda - \lambda_0| < \epsilon$$

the equation

$$(I - A(\lambda))\,\phi = 0 \tag{5.1}$$

has the same number of linearly independent solutions.

PROOF. Let us denote by $e_1, e_2, \cdots, e_n$ an orthonormal basis of the space of all solutions of the equation $(I - A(\lambda_0))\,\phi = 0$, and by $g_1, g_2, \cdots, g_n$ an orthonormal basis of the subspace $[\,\Re(I - A(\lambda_0))\,]^{\perp}$. We form the operator $B(\lambda)$, putting

$$B(\lambda) = I - A(\lambda) + \sum_{j=1}^{n} (\cdot, e_j)\, g_j \quad (\lambda \in U_{\lambda_0}).$$

It is easily seen that the operator $B(\lambda_0)$ does not annihilate any nonzero vector, and that its range coincides with the entire space $\mathfrak{H}$. Hence the operator $B(\lambda)$ is invertible for $\lambda = \lambda_0$. But then there exists a suf-

ficiently small positive number ρ such that the operator $B(\lambda)$ is invertible for all points λ from the disk $|\lambda - \lambda_0| < \rho$:

$$\begin{aligned} B^{-1}(\lambda) &= (B(\lambda_0) + (B(\lambda) - B(\lambda_0)))^{-1} \\ &= B^{-1}(\lambda_0)(I + (B(\lambda) - B(\lambda_0))B^{-1}(\lambda_0))^{-1} \\ &= B^{-1}(\lambda_0)\left(I + \sum_{k=1}^{\infty}[(B(\lambda_0) - B(\lambda))B^{-1}(\lambda_0)]^k\right). \end{aligned}$$

The equation $(I - A(\lambda))\phi = 0$ is obviously equivalent to the equation

$$B(\lambda)\phi = \sum_{j=1}^{n}(\phi, e_j)g_j$$

or to the system of equations

$$\phi = \sum_{j=1}^{n}\xi_j B^{-1}(\lambda)g_j \quad (|\lambda - \lambda_0| < \rho) \tag{5.2}$$

$$\xi_k = (\phi, e_k) \quad (k = 1, 2, \cdots, n). \tag{5.3}$$

Inserting in (5.3) the expression for ϕ from (5.2), we obtain for the determination of the numbers ξ_j $(j = 1, 2, \cdots, n)$ a homogeneous system of n linear algebraic equations:

$$\sum_{j=1}^{n}[\delta_{jk} - (B^{-1}(\lambda)g_j, e_k)]\xi_j = 0 \qquad (k = 1, 2, \cdots, n). \tag{5.4}$$

For $|\lambda - \lambda_0| < \rho$ the number of linearly independent solutions of the equation (5.1) coincides with the number of linearly independent solutions of the system (5.4).

All the elements of the determinant $\Delta(\lambda)$ of the system (5.4) are analytic functions of the parameter λ in the disk $|\lambda - \lambda_0| < \rho$. If all of them are identically equal to zero, then the system (5.4) has, for all λ from the disk $|\lambda - \lambda_0| < \rho$, exactly n linearly independent solutions. In this case the lemma is proved.

Suppose that at least one element of the determinant $\Delta(\lambda)$ is different from zero at some point of the disk $|\lambda - \lambda_0| < \rho$. We denote by $\Delta_p(\lambda)$ any minor of highest order among those minors of the determinant $\Delta(\lambda)$ which are different from zero at one point at least of the disk $|\lambda - \lambda_0| < \rho$; the index p indicates the order of this minor. By virtue of the analyticity of $\Delta_p(\lambda)$, we will have $\Delta_p(\lambda) \neq 0$ for all λ of the disk under consideration with the possible exception of certain isolated points. At all points λ for which the determinant $\Delta_p(\lambda)$ does not vanish the system (5.4) will have $n - p$ linearly independent solutions.

There is a largest disk $|\lambda - \lambda_0| < \epsilon$, at all of whose points, except possibly $\lambda = \lambda_0$, one has $\Delta_p(\lambda) \neq 0$. For all λ satisfying the inequality $0 < |\lambda - \lambda_0| < \epsilon$, the system (5.4) will have exactly $n - p$ linearly independent solutions.

The lemma is proved.

THEOREM 5.1 (I. C. GOHBERG [1]). *Suppose that $A(\lambda)$ is an operator-function, holomorphic in an open connected set G, with values from $\mathfrak{S}_\infty$. Then for all points $\lambda \in G$, with the possible exception of certain isolated points, the number $\alpha(\lambda)$ of linearly independent solutions of the equation*

$$\phi - A(\lambda)\phi = 0$$

is constant:

$$\alpha(\lambda) = n;$$

at the isolated points mentioned,

$$\alpha(\lambda) > n.$$

In particular, if $\alpha(\lambda) = 0$ for at least one point, then for all $\lambda \in G$, with the possible exception of certain isolated points, the operator $I - A(\lambda)$ has a bounded inverse.

PROOF. Suppose that $n = \min \alpha(\lambda)$ $(\lambda \in G)$ and that this minimum is attained at the point $\lambda = \lambda_0$, i.e., $\alpha(\lambda_0) = n$.

We denote by λ_1 any point from G, for which $\alpha(\lambda_1) > n$. We shall show that λ_1 is an isolated point, i.e., we can find $\epsilon_1 > 0$ such that for all λ in the punctured disk $0 < |\lambda - \lambda_1| < \epsilon_1$ we will have $\alpha(\lambda) = n$. To this end, we connect the points λ_0 and λ_1 by a curve Γ lying entirely in G. Applying Lemma 5.1 to the operator $A(\lambda)$, we find that to every point λ of the curve Γ there corresponds a number $\epsilon_\lambda > 0$ such that, for all $\tilde{\lambda}$ satisfying the inequality $0 < |\tilde{\lambda} - \lambda| < \epsilon_\lambda$, the function $\alpha(\tilde{\lambda})$ has a fixed value. Constructing such a neighborhood U_λ for every point $\lambda \in \Gamma$, we obtain a covering of Γ. Let us select from this covering a finite subcovering $U_1, U_2, \cdots, U_N$ $(\lambda_0 \in U_1; \lambda_1 \in U_N)$.

Noting that adjacent neighborhoods of this covering intersect, we conclude that at all points of the neighborhoods U_j $(j = 1, 2, \cdots, N)$, with the possible exception of their centers, the function $\alpha(\lambda)$ has one fixed value. But since, in the neighborhood U_1 which contains the point λ_0, we have $\alpha(\lambda) = n$, it follows that also $\alpha(\lambda) = n$ over the entire neighborhood U_N of the point λ_1, except at the point λ_1 itself.

This proves the first assertion of the theorem. The second assertion follows at once from the first, if we take into consideration that at all

points λ of the region G for which $\alpha(\lambda) = 0$, the operator $I - A(\lambda)$ has a bounded inverse.

We need yet another lemma.

LEMMA 5.2. *Let $A \in \mathfrak{R}$, and let G be any connected component of the set $\tilde{\rho}(A)$ of all normal points of the operator A.*

If $B \in \mathfrak{S}_\infty$ and if the operator $A + B$ has at least one regular point in G, then G is a connected component of $\tilde{\rho}(A + B)$.

In particular, for any $A \in \mathfrak{R}$ and $B \in \mathfrak{S}_\infty$, the unbounded components of $\tilde{\rho}(A)$ and $\tilde{\rho}(A + B)$ coincide.

PROOF. For any $\lambda \in \rho(A) \cap G$

$$A + B - \lambda I = (I + B(A - \lambda I)^{-1})(A - \lambda I).$$

The operator-function $B(A - \lambda I)^{-1}$ is holomorphic in the region $\rho(A) \cap G$ and assumes values from $\mathfrak{S}_\infty$. Since the operator $A + B - \lambda I$ is invertible at some point of G, then at this point the operator $I + B(A - \lambda I)^{-1}$ is invertible. Hence by Theorem 5.1 the operator $I + B(A - \lambda I)^{-1}$ is invertible everywhere in $\rho(A) \cap G$, with the possible exception of a finite or countable set of isolated points. Since moreover the spectrum of A in G consists of isolated points, the spectrum of $A + B$ in G likewise consists of isolated points.

Let us denote by Γ an arbitrary rectifiable simple closed contour, consisting of regular points of the operators A and $A + B$ and enclosing some region entirely contained in G.

We have

$$(A + B - \lambda I)^{-1} = (A - \lambda I)^{-1} - (A + B - \lambda I)^{-1}B(A - \lambda I)^{-1} \quad (\lambda \in \Gamma),$$

and consequently

$$-\frac{1}{2\pi i}\int_\Gamma (A + B - \lambda I)^{-1} d\lambda = -\frac{1}{2\pi i}\int_\Gamma (A - \lambda I)^{-1} d\lambda$$

$$+\frac{1}{2\pi i}\int_\Gamma (A + B - \lambda I)^{-1}B(A - \lambda I)^{-1} d\lambda. \tag{5.5}$$

The first term on the right side of this equality is, by Theorem 2.2, a finite-dimensional projector, and the second term is a completely continuous operator, since $B \in \mathfrak{S}_\infty$.

It follows that the projector represented by the left side of (5.5) is completely continuous. This is possible only when it is finite-dimensional. But then, by Theorem 2.2, all the points of the spectrum of the operator $A + B$ which are encircled by the contour Γ are normal eigenvalues of this operator.

To complete the proof of the lemma, it remains to note that the roles of the operators A and $A+B$ can be interchanged and that the intersection of $\tilde{\rho}(A)$ and $\tilde{\rho}(A+B)$ always contains the unbounded region $|\lambda| > \max(|A|, |A+B|)$.

As a simple corollary of the lemma just proved, we obtain the following result.

THEOREM 5.2. *Let H be a selfadjoint operator from $\mathfrak{R}$, and B, any operator from $\mathfrak{S}_\infty$. Then the set of normal points of the operators H and $H+B$ coincide. Consequently, every nonreal λ is either a regular point or a normal eigenvalue for $H+B$.*[7]

PROOF. As is well known, the region $\rho(H)$ contains all nonreal points and all real points λ for which $|\lambda| > |H|$, and thus is connected. Therefore Theorem 5.2 follows from Lemma 5.2.

By the same means, one can prove a theorem on perturbations of a unitary operator.

THEOREM 5.3. *Let $A = U + B$, where U is a unitary operator and $B \in \mathfrak{S}_\infty$. Then, if the operator A has at least one regular point in the disk $|\lambda| < 1$, the sets of normal points of the operators A and U coincide.*

Hence in this case any point λ with $|\lambda| \neq 1$ is either a regular point or a normal eigenvalue of the operator A.

From Theorem 5.2 there follows an important result.

THEOREM 5.4. *Suppose that the spectrum of the operator $A \in \mathfrak{R}$ consists of the single point $\lambda = 0$.*

If one of the Hermitian components of the operator A,

$$A_{\mathscr{R}} = (A + A^*)/2 \quad or \quad A_{\mathscr{J}} = (A - A^*)/2i$$

is completely continuous, then the second is completely continuous, and, consequently, A is a Volterra operator.

PROOF. For definiteness, we assume that $A_{\mathscr{J}} \in \mathfrak{S}_\infty$.

By Theorem 5.2, all nonzero points of the spectrum of the operator

$$A_{\mathscr{R}} = A - iA_{\mathscr{J}}$$

are normal points of it. Since the operator $A_{\mathscr{R}}$ is, moreover, selfadjoint, it is completely continuous (cf. N. I. Ahiezer and I. M. Glazman [1]). The theorem is proved.

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[7] Theorem 5.2 admits of generalization to the case where H and B are unbounded operators (cf. Lemma V.10.1).

CHAPTER II

s-NUMBERS OF COMPLETELY CONTINUOUS OPERATORS

s-numbers were apparently first introduced by E. Schmidt in the study of integral equations with nonsymmetric (non-hermitean) kernels. But more than three decades passed before the important properties of the s-numbers of matrices and completely continuous operators, expressed in the form of various inequalities, were established. Still later it was found that these properties of s-numbers play an important role in the construction of a general theory of symmetrically normed ideals of completely continuous operators (cf. Chapter III), and also for the study of the asymptotic properties of spectra and in other questions.

This chapter contains a rather complete account of the basic properties of s-numbers of operators. A number of results concerning s-numbers are perhaps presented here for the first time.

§1. Minimax properties of the eigenvalues of selfadjoint completely continuous operators

In studying the s-numbers of completely continuous operators, repeated use is made of the well-known minimax properties of the eigenvalues of completely continuous operators.

For the convenience of the reader we shall in this section cite without proof a theorem describing these properties, and indicate various consequences of it.

Let A be any linear selfadjoint completely continuous operator. Then, as is well known, all of its eigenvalues are real, and every root vector of the operator A is an eigenvector of A. The operator A admits a uniformly convergent representation

$$A = \sum_{j=1}^{\nu(A)} \lambda_j(A)\,(\cdot, \phi_j)\,\phi_j, \tag{1.1}$$

where ϕ_j $(j = 1, 2, \cdots, \nu(A))$ is an orthonormal system of eigenvectors of A, complete in $\mathfrak{R}(A)$, such that

$$A\phi_j = \lambda_j(A)\phi_j \qquad (j = 1, 2, \cdots, \nu(A)).$$

We shall say that a bounded linear operator A is *nonnegative* and write $A \geqq 0$, if

$$(Af,f) \geqq 0$$

for every $f \in \mathfrak{H}$. As is well known, every nonnegative operator is selfadjoint.

A selfadjoint completely continuous operator will be nonnegative if and only if all of its eigenvalues are nonnegative.

One has the following result (cf. Riesz and Sz.-Nagy [1] §95).

THEOREM (ON THE MINIMAX PROPERTIES OF EIGENVALUES).[1] *Let A ($\neq 0$) be a nonnegative completely continuous operator and let ϕ_j ($j = 1, 2, \cdots$) be an orthonormal system of its eigenvectors which is complete in $\mathfrak{R}(A)$, so that*

$$A\phi_j = \lambda_j(A)\phi_j \qquad (j = 1, 2, \cdots),$$

where $\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots$. Then its eigenvalues have the following minimax properties.

1)

$$\lambda_1(A) = \max_{\phi \in \mathfrak{H}} \frac{(A\phi, \phi)}{(\phi, \phi)}, \tag{1.2}$$

where the maximum in (1.2) *is attained only for those eigenvectors of the operator A which correspond to $\lambda_1(A)$.*

2)

$$\lambda_{j+1}(A) = \min_{\mathfrak{L} \in \mathfrak{N}_j} \max_{\phi \in \mathfrak{L}^\perp} \frac{(A\phi, \phi)}{(\phi, \phi)} \qquad (j = 1, 2, \cdots), \tag{1.3}$$

where $\mathfrak{N}_j$ ($1 \leqq j < \nu(A)$) is the set of all j-dimensional lineals of the space $\mathfrak{H}$, and the minimum in (1.3) *is attained when $\mathfrak{L}$ coincides with the linear hull $\mathfrak{L}_j$ of the eigenvectors $\phi_1, \phi_2, \cdots, \phi_j$, so that*

$$\lambda_{j+1}(A) = \max_{\phi \in \mathfrak{L}_j^\perp} \frac{(A\phi, \phi)}{(\phi, \phi)}. \tag{1.4}$$

We remark that even in the case where $\lambda_{j+1} < \lambda_j$, the minimum in (1.3) can be attained for $\mathfrak{L} \neq \mathfrak{L}_j$ (cf. M. D. Dol′berg [2]).

From the assertion 1) it easily follows that the maximum in (1.4) is attained only at the eigenvectors of the operator A corresponding to the eigenvalue $\lambda_{j+1}(A)$.

With the help of the relations (1.2)—(1.4), one can easily prove the following lemma.

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[1] As is well known, this result can be extended to the *positive* eigenvalues $\lambda_1^+(A) \geqq \lambda_2^+(A) \geqq \cdots$ of *any* completely continuous selfadjoint operator.

LEMMA 1.1. *Let* $A, B \in \mathfrak{S}_\infty$ *and* $0 \leq A \leq B$;[2] *then*

$$\lambda_j(A) \leqq \lambda_j(B) \qquad (j = 1, 2, \cdots).$$

In order that equality hold simultaneously in all of these relations, it is necessary and sufficient that $A = B$.

It follows in an obvious way from Lemma 1.1 that $\nu(A) \leqq \nu(B)$ if $0 \leqq A \leqq B$.

Let A be a completely continuous selfadjoint operator with spectral decomposition (1.1). We form the nonnegative operators

$$A_+ = \sum_{\lambda_j > 0} \lambda_j(A)(\cdot, \phi_j)\phi_j; \qquad A_- = -\sum_{\lambda_j < 0} \lambda_j(A)(\cdot, \phi_j)\phi_j.$$

Obviously $A = A_+ - A_-$.

LEMMA 1.2. *Suppose that the selfadjoint operator* $A \in \mathfrak{S}_\infty$ *can be represented as the difference*

$$A = H_1 - H_2$$

of nonnegative operators $H_1, H_2 \in \mathfrak{S}_\infty$. *Then*

$$\lambda_j(A_+) \leqq \lambda_j(H_1) \quad \textit{and} \quad \lambda_j(A_-) \leqq \lambda_j(H_2) \quad (j = 1, 2, \cdots). \tag{1.5}$$

PROOF. In fact, we have

$$(H_1 f, f) = (Af, f) + (H_2 f, f) \geqq (Af, f) \quad (f \in \mathfrak{H})$$

and

$$(H_2 f, f) \geqq -(Af, f) \quad (f \in \mathfrak{H}).$$

The relations (1.5) immediately follow from this, on the basis of the minimax properties of eigenvalues (cf. the footnote to the theorem on the minimax properties of eigenvalues).

§2. s-numbers of completely continuous operators and their simplest properties

1. s-numbers of a completely continuous operator. Let $A \in \mathfrak{S}_\infty$; then $H = (A^*A)^{1/2} \in \mathfrak{S}_\infty$. As the first definition of s-numbers we take the following:

The eigenvalues of the operator H are called the *s-numbers* of the operator A.

We shall enumerate the nonzero s-numbers in decreasing order,

[2] The inequality $A \leq B$ signifies that the operator $B - A$ is nonnegative.

taking account of their multiplicities, so that

$$s_j(A) = \lambda_j(H) \quad (j = 1, 2, \cdots, r(H)).^{3)}$$

If $r(H) < \infty$ we shall put

$$s_j(A) = 0 \quad \text{for} \quad j = r(H) + 1, \cdots.$$

We note that

$$s_1(A) = |A|.$$

If the operator $A \in \mathfrak{S}_\infty$ is selfadjoint or at least normal (i.e. if A and A^* commute), then

$$s_j(A) = |\lambda_j(A)| \quad (j = 1, 2, \cdots).$$

It is also obvious that for any scalar c

$$s_j(cA) = |c| s_j(A) \quad (j = 1, 2, \cdots).$$

The following two important properties of the s-numbers of a completely continuous operator are somewhat less trivial.

I)

$$s_j(A) = s_j(A^*) \quad (j = 1, 2, \cdots). \tag{2.1}$$

II) *For any bounded operator B,*

$$s_j(BA) \leqq |B| s_j(A) \quad (j = 1, 2, \cdots), \tag{2.2}$$

$$s_j(AB) \leqq |B| s_j(A) \quad (j = 1, 2, \cdots). \tag{2.3}$$

The first property will be obtained in §2.2. Let us prove the second. By definition

$$s_j^2(BA) = \lambda_j(A^*B^*BA), \quad s_j^2(A) = \lambda_j(A^*A) \quad (j = 1, 2, \cdots).$$

On the other hand,

$$(A^*B^*BAf, f) = |BAf|^2 \leqq |B|^2 |Af|^2 = |B|^2 (A^*Af, f) \quad (f \in \mathfrak{H}),$$

so that $A^*B^*BA \leqq |B|^2 A^*A$, from which, on the basis of Lemma 1.1, we obtain the relations (2.2).

Since, by virtue of (2.1), $s_j(AB) = s_j(B^*A^*)$, and according to what was just proved $s_j(B^*A^*) \leqq |B^*| s_j(A^*) = |B| s_j(A)$, the relations (2.3) are also established.

Using Lemma 1.1, it is not hard to show that equality holds in (2.2)

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3) We recall that $r(H) = \dim \mathfrak{R}(H)$.

(in (2.3)) simultaneously for all $j = 1, 2, \cdots$ if and only if the operator $B/|B|$ $(B^*/|B^*|)$ is isometric on the set $\Re(A)$ $(\Re(A^*))$.

2. The Schmidt expansion of a completely continuous operator. Let A be a completely continuous operator and let $A = UH$ be its polar representation.

We denote by ϕ_j $(j = 1, 2, \cdots, r(H))$ an orthonormal system of eigenvectors of the operator H which is dense in $\Re(H)$. Then

$$H = \sum_{j=1}^{r(H)} s_j(A)(\cdot, \phi_j)\phi_j, \tag{2.4}$$

where the series on the right side converges in the uniform operator norm. Applying the operator U to both sides of (2.4), we obtain

$$A = \sum_{j=1}^{r(A)} s_j(A)(\cdot, \phi_j)\, U\phi_j.$$

Since $\phi_j \in \Re(H)$, the system $U\phi_j$ $(j = 1, 2, \cdots, r(A))$ is orthonormal.

Thus every linear completely continuous operator A admits a Schmidt expansion

$$A = \sum_{j=1}^{r(A)} s_j(A)(\cdot, \phi_j)\psi_j, \tag{2.5}$$

where $\{\phi_j\}$ and $\{\psi_j\}$ are certain orthonormal systems, and the series in (2.5), just as does the series in (2.4), converges in the uniform norm.

It follows from (2.5) that

$$A^* = \sum_{j=1}^{r(A)} s_j(A)(\cdot, \psi_j)\phi_j. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$A^*A\phi_j = s_j^2(A)\phi_j \text{ and } AA^*\psi_j = s_j^2(A)\psi_j \qquad (j = 1, 2, \cdots, r(A)).$$

Hence we conclude that

$$s_j(A) = s_j(A^*) \qquad (j = 1, 2, \cdots),$$

i.e. the relations (2.1) hold.

3. An approximation property of s-numbers, and resulting inequalities. Henceforth we shall denote by $\mathfrak{K}_n$ $(n = 0, 1, 2, \cdots)$ the set of all finite-dimensional operators of dimension $\leq n$.

THEOREM 2.1 (DŽ. È. ALLAHVERDIEV [2]). *Let A be a linear completely continuous operator; then for any $n = 0, 1, 2, \cdots$*

$$s_{n+1}(A) = \min_{K \in \mathfrak{K}_n} |A - K|. \tag{2.7}$$

PROOF. Let K be a linear n-dimensional operator. Obviously the subspace $\mathfrak{H} \ominus \mathfrak{Z}(K)$ is n-dimensional; consequently, by virtue of the minimax properties of eigenvalues,

$$s_{n+1}(A) \leqq \max_{\phi \in \mathfrak{Z}(K)} \frac{|A\phi|}{|\phi|}.$$

Since for all $\phi \in \mathfrak{Z}(K)$

$$|A\phi| = |(A - K)\phi| \leqq |A - K|\,|\phi|,$$

it follows that

$$s_{n+1}(A) \leqq |A - K|.$$

To complete the proof, it remains to note that

$$|A - K_n| = s_{n+1}(A),$$

where

$$K_n = \sum_{j=1}^{n} s_j(A)(\cdot, \phi_j)\psi_j \tag{2.8}$$

is the nth partial sum of the Schmidt expansion (2.5) of the operator A.

The formula (2.7) shows that $s_{n+1}(A)$ is the distance from the operator A to the set $\mathfrak{K}_n$. This result can be taken as a new, equivalent definition of the s-numbers. For many questions this definition turns out to be more convenient than the original one.

COROLLARY 2.1. *Let $A \in \mathfrak{S}_\infty$ and let T be any r-dimensional operator. Then*

$$s_{n+r}(A) \leqq s_n(A + T) \leqq s_{n-r}(A). \tag{2.9}$$

In fact, let K_n be the operator defined by (2.8); then by Theorem 2.1,

$$s_{n+1}(A) = |(A + T) - (T + K_n)| \geqq s_{n+r+1}(A + T) \tag{2.10}$$
$$(n = 0, 1, \cdots).$$

Interchanging the roles of the operators A and $A + T$, we obtain

$$s_{n+1}(A + T) \geqq s_{n+r+1}(A) \qquad (n = 0, 1, \cdots). \tag{2.11}$$

From (2.10) and (2.11) follows (2.9).

COROLLARY 2.2 (K. FAN [3]). *If $A, B \in \mathfrak{S}_\infty$, then*

$$(2.12)\qquad s_{m+n-1}(A+B) \leqq s_m(A) + s_n(B) \qquad (m,n = 1,2,\cdots)$$

and

$$(2.13)\qquad s_{m+n-1}(AB) \leqq s_m(A)s_n(B) \qquad (m,n = 1,2,\cdots).$$

In fact, let the $(m-1)$-dimensional operator K_1 and the $(n-1)$-dimensional operator K_2 be such that

$$s_m(A) = |A - K_1| \quad\text{and}\quad s_n(B) = |B - K_2|.$$

Then

$$\begin{aligned} s_{m+n-1}(A+B) &\leqq |A + B - (K_1 + K_2)| \\ &\leqq |A - K_1| + |B - K_2| \leqq s_m(A) + s_n(B). \end{aligned}$$

Moreover, since the dimension of the operator $AK_2 + K_1(B - K_2)$ does not exceed $m + n - 2$, and $(A - K_1)(B - K_2) = AB - AK_2 - K_1(B - K_2)$, we have

$$s_{m+n-1}(AB) \leqq |A - K_1|\,|B - K_2|,$$

from which (2.13) follows. Special cases of the relations (2.12) and (2.13) are the well-known inequalities of H. Weyl for nonnegative completely continuous operators H_1 and H_2:

$$\lambda_{n+m-1}(H_1 + H_2) \leqq \lambda_n(H_1) + \lambda_m(H_2).$$

These inequalities can be generalized to the case of any selfadjoint operators from $\mathfrak{S}_\infty$ (cf. Riesz and Sz.-Nagy [1] §95).

COROLLARY 2.3. *For any operators* $A, B \in \mathfrak{S}_\infty$,

$$|s_n(A) - s_n(B)| \leqq |A - B| \qquad (n = 1,2,\cdots).$$

In fact,

$$\begin{aligned} s_{n+1}(A) &= \min_{K\in\mathfrak{K}_n} |A - K| = \min_{K\in\mathfrak{K}_n} |B - K + A - B| \\ &\leqq \min_{K\in\mathfrak{K}_n} |B - K| + |A - B| = s_{n+1}(B) + |A - B|. \end{aligned}$$

Interchanging the roles of the operators A and B, we obtain

$$s_{n+1}(B) \leqq s_{n+1}(A) + |A - B|,$$

from which follows the relation to be established.

4. Geometric interpretation of s-numbers. Closely related to the approximation theorem (Theorem 2.1) is a theorem which gives a geometric

approximation interpretation of the s-numbers of an operator $A \in \mathfrak{S}_\infty$. In this interpretation use is made of the concept of the nth width of a set, which A. N. Kolmogorov [1] (cf. also V. M. Tihomirov [1]) has defined for any set $\mathscr{E}$ belonging to an arbitrary linear metric space M. We shall present a definition of this concept for the case of interest to us, in which the space M is a Hilbert space $\mathfrak{H}$, and $\mathscr{E}$ is some centrally-symmetric set ($\mathscr{E} = -\mathscr{E}$, i.e., if $x \in \mathscr{E}$, then $-x \in \mathscr{E}$).

Let us denote by $\mathfrak{P}_n$ the collection of all n-dimensional orthoprojectors P, acting in $\mathfrak{H}$. Then the *nth width* ($n = 1, 2, \cdots$) of the set $\mathscr{E}$ is the number $d_n(\mathscr{E})$ ($\leqq \infty$) defined by

$$d_n(\mathscr{E}) = \inf_{P \in \mathfrak{P}_n} \sup_{x \in \mathscr{E}} |x - Px|.$$

Every orthoprojector $P \in \mathfrak{P}_n$ is completely defined by the n-dimensional lineal $\mathfrak{L}_n$ on which it projects $\mathfrak{H}$; the quantity $|x - Px|$ gives the distance from the point x to $\mathfrak{L}_n$, and

$$\delta(\mathscr{E}, \mathfrak{L}_n) = \sup_{x \in \mathscr{E}} |x - Px|$$

gives the deviation of $\mathscr{E}$ from $\mathfrak{L}_n$. Thus the width $d_n(\mathscr{E})$ is the infimum of the deviation of the set $\mathscr{E}$ from the n-dimensional lineals in $\mathfrak{H}$. In those cases where this infimum is attained, there will exist an n-dimensional lineal $\mathfrak{L}_n^{(0)}$ which deviates least from $\mathscr{E}$, and the number $d_n(\mathscr{E})$ will be the deviation of $\mathscr{E}$ from $\mathfrak{L}_n^{(0)}$.

THEOREM 2.2. *Let A be a completely continuous operator. Then $s_{n+1}(A)$ ($n = 1, 2, \cdots$) coincides with the nth width of the set $\mathscr{E} = A\mathscr{S}$, onto which the operator A maps the unit ball $\mathscr{S}$ ($|x| \leqq 1$) in $\mathfrak{H}$.*

This theorem enables us to give a third definition of the s-numbers which is equivalent to the first two.

PROOF. If $P \in \mathfrak{P}_n$ then $PA \in \mathfrak{K}_n$. Thus for any $P \in \mathfrak{P}_n$ and $K = PA$ we have

$$\sup_{x \in \mathscr{E}} |x - Px| = \sup_{|y| \leq 1} |Ay - PAy| = |A - K| \geqq s_{n+1}(A),$$

from which $d_n(\mathscr{E}) \geqq s_{n+1}(A)$.

Since, on the other hand, for

$$P = \sum_{k=1}^{n} (\cdot, \psi_k)\psi_k,$$

where the ψ_k ($k = 1, 2, \cdots, n$) are taken from the Schmidt expansion

(2.5) of the operator A, the operator $K = PA$ has the form (2.8), and $|A - K| = s_{n+1}(A)$, we have $d_n(\mathscr{C}) = s_{n+1}(A)$.

The theorem is proved.

We have at the same time proved that the lineal $\mathfrak{L}_n$, spanned by the vectors $\psi_1, \psi_2, \cdots, \psi_n$ $(n = 1, 2, \cdots)$, is an n-dimensional lineal having minimum deviation from the set $\mathscr{C} = A\mathscr{S}$.

5. An asymptotic theorem concerning s-numbers.

THEOREM 2.3 (K. FAN [3]). *Suppose $A, B \in \mathfrak{S}_\infty$ and that for some $r > 0$*

$$\lim_{n\to\infty} n^r s_n(A) = a \quad \text{and} \quad \lim_{n\to\infty} n^r s_n(B) = 0. \tag{2.14}$$

Then

$$\lim_{n\to\infty} n^r s_n(A + B) = a. \tag{2.15}$$

PROOF. From (2.12) it follows that

$$s_{(k+1)m+j}(A + B) \leqq s_{km+j}(A) + s_{m+1}(B) \quad (m = 1, 2, \cdots; j = 0, 1, \cdots, k).$$

Since any integer n admits the representation $n = (k+1)m + j$, it follows that

$$\overline{\lim}\, n^r s_n(A + B) \leqq ((k+1)/k)^r a$$

and therefore, since k is arbitrary,

$$\overline{\lim}\, n^r s_n(A + B) \leqq a. \tag{2.16}$$

On the other hand, $A = (A + B) - B$ and consequently

$$s_{(k+1)m+j}(A) \leqq s_{km+j}(A + B) + s_{m+1}(B),$$

or

$$s_{km+j}(A + B) \geqq s_{(k+1)m+j}(A) - s_{m+1}(B).$$

Setting $n = km + j$ and letting n go to infinity, we obtain

$$\underline{\lim}\, n^r s_n(A + B) \geqq (k/(k+1))^r a,$$

or, since k is arbitrary,

$$\underline{\lim}\, n^r s_n(A + B) \geqq a. \tag{2.17}$$

From (2.16) and (2.17) follows (2.15).

The theorem is proved.

This theorem can be immediately generalized to the case where the function n^r in (2.14) and (2.15) is replaced by the function $\phi_r(n) = n^r L(n)$,

where $L(\nu)$ is an arbitrary positive function with the property:

$$\lim L(\nu_2)/L(\nu_1) = 1,$$

for $\nu_1 \to \infty$, $\nu_2 \to \infty$ and $\nu_2/\nu_1 \to b$, for every choice of b $(0 < b < \infty)$. This remark will be needed further on.

§3. Inequalities relating the s-numbers, eigenvalues and diagonal elements of completely continuous operators

1. In this subsection we shall present, with certain changes in their proofs, the results of H. Weyl from his concise original communication [2].

LEMMA 3.1 (H. WEYL [2], A. HORN [1]). *Let A be a linear completely continuous operator and $s_j = s_j(A)$ $(j = 1, 2, \cdots)$. Then*

$$\det \| (A\phi_j, A\phi_k) \|_1^n \leq s_1^2 s_2^2 \cdots s_n^2 \det \| (\phi_j, \phi_k) \|_1^n \tag{3.1}$$

for any system of vectors $\phi_1, \phi_2, \cdots, \phi_n$.

PROOF. Let us denote by e_j $(j = 1, 2, \cdots)$ a complete orthonormal system of eigenvectors of the operator A^*A. Then

$$(A\phi_j, A\phi_k) = (A^*A\phi_j, \phi_k) = \sum_{l=1}^{\infty} s_l^2 (\phi_j, e_l)(e_l, \phi_k).$$

Therefore the square matrix $\mathscr{A} = \| (A\phi_j, A\phi_k) \|_1^n$ can be represented in the form $\mathscr{A} = \mathscr{B}\mathscr{B}^*$, where $\mathscr{B}$ is a rectangular matrix:

$$\mathscr{B} = \| s_r (\phi_j, e_r) \|_{\substack{j=1,2,\cdots,n \\ r=1,2,\cdots}}.$$

By the Binet-Cauchy theorem from the theory of determinants,

$$\det \mathscr{A} = \sum_{1 \leq r_1 < r_2 < \cdots < r_n < \infty} \mathscr{B} \begin{pmatrix} 1\ 2 \cdots n \\ r_1 r_2 \cdots r_n \end{pmatrix} \mathscr{B}^* \begin{pmatrix} r_1 r_2 \cdots r_n \\ 1\ 2 \cdots n \end{pmatrix}, \tag{3.2}$$

where

$$\mathscr{B} \begin{pmatrix} 1\ 2 \cdots n \\ r_1 r_2 \cdots r_n \end{pmatrix}$$

denotes the minor of the matrix $\mathscr{B}$, consisting of the intersections of rows $1, 2, \cdots, n$ with columns $r_1, r_2, \cdots, r_n$.

Let us denote by $\mathscr{C}$ the matrix

$$\| (\phi_j, e_r) \|_{\substack{j=1,2,\cdots,n \\ r=1,2,\cdots}}.$$

Obviously

$$\mathscr{B}\begin{pmatrix}1\ 2\cdots n\\ r_1r_2\cdots r_n\end{pmatrix} = s_{r_1}s_{r_2}\cdots s_{r_n}\mathscr{C}\begin{pmatrix}1\ 2\cdots n\\ r_1r_2\cdots r_n\end{pmatrix}$$

and consequently by (3.2)

$$\det \mathscr{A} \leqq s_1^2 s_2^2\cdots s_n^2 \sum_{r_1,r_2,\cdots,r_n} \mathscr{C}\begin{pmatrix}1\ 2\cdots n\\ r_1r_2\cdots r_n\end{pmatrix}\mathscr{C}^*\begin{pmatrix}r_1r_2\cdots r_n\\ 1\ 2\cdots n\end{pmatrix}.$$

Since

$$\sum_{r_1,r_2,\cdots,r_n} \mathscr{C}\begin{pmatrix}1\ 2\cdots n\\ r_1r_2\cdots r_n\end{pmatrix}\mathscr{C}^*\begin{pmatrix}r_1r_2\cdots r_n\\ 1\ 2\cdots n\end{pmatrix}$$

$$= \det\left\| \sum_{r=1}^{\infty}(\phi_j,e_r)(e_r,\phi_k)\right\|_1^n = \det\|(\phi_j,\phi_k)\|_1^n,$$

we see that (3.1) is valid.

LEMMA 3.2. *Let A be any linear completely continuous operator, and ϕ_j $(j=1,2,\cdots,r(A))$ some orthonormal system. If the equalities*

$$|(A\phi_j,\phi_j)| = s_j(A) \qquad (j=1,2,\cdots,r(A)), \tag{3.3}$$

are fulfilled, then the operator A is normal, and the ϕ_j $(j=1,2,\cdots,r(A))$ form a complete system of its eigenvectors in $\overline{\mathfrak{R}(A)}$.

PROOF. In fact, since

$$s_1^2(A) = |(A\phi_1,\phi_1)|^2 \leqq |A\phi_1|^2 = (A^*A\phi_1,\phi_1)$$

and

$$s_1^2(A) = \max_{|\phi|=1}(A^*A\phi,\phi),$$

it follows that

$$s_1^2(A) = (A^*A\phi_1,\phi_1).$$

By the minimax properties of the eigenvalues, the last equality shows that

$$A^*A\phi_1 = s_1^2(A)\phi_1.$$

From the relations

$$s_2^2(A) = |(A\phi_2,\phi_2)|^2 \leqq |A\phi_2|^2 = (A^*A\phi_2,\phi_2)$$

and

$$s_2^2(A) = \max_{|\phi|=1,\,(\phi,\phi_1)=0}(A^*A\phi,\phi)$$

we obtain

$$A^*A\phi_2 = s_2^2(A)\phi_2.$$

Continuing the argument in the same way, we obtain

$$A^*A\phi_j = s_j^2(A)\phi_j \qquad (j = 1, 2, \cdots, r(A));$$

consequently the ϕ_j $(j = 1, 2, \cdots, r(A))$ form a complete system of eigenvectors of the operator A^*A, corresponding to its nonzero eigenvalues. It follows that the operator A^*A, and along with it the operator A, vanishes on the subspace $\mathfrak{L}$ orthogonal to all of the vectors ϕ_j $(j = 1, 2, \cdots, r(A))$.

By the same reasoning one establishes that the operator A^* likewise vanishes on the subspace $\mathfrak{L}$. It follows that any vector $f \in \overline{\mathfrak{R}(A)}$ can be represented in the form

$$f = \sum_{j=1}^{r(A)} (f, \phi_j)\phi_j;$$

in particular,

$$A\phi_j = \sum_{k=1}^{r(A)} (A\phi_j, \phi_k)\phi_k. \tag{3.4}$$

Consequently,

$$(A\phi_j, A\phi_j) = \sum_{k=1}^{r(A)} |(A\phi_j, \phi_k)|^2. \tag{3.5}$$

Since

$$|(A\phi_j, \phi_j)|^2 = s_j^2(A) = (A\phi_j, A\phi_j),$$

it follows from (3.5) that

$$(A\phi_j, \phi_k) = 0 \qquad (j \neq k).$$

Thus, by (3.4),

$$A\phi_j = (A\phi_j, \phi_j)\phi_j \qquad (j = 1, 2, \cdots, r(A)),$$

and so the operator A can be represented in the form

$$A = \sum_{j=1}^{r(A)} (A\phi_j, \phi_j)(\cdot, \phi_j)\phi_j.$$

The stated properties of the operator A follow at once.

LEMMA 3.3 (H. WEYL [2]). *Let A be any linear completely continuous operator. Then*

$$|\lambda_1(A)\lambda_2(A)\cdots\lambda_n(A)| \leqq s_1(A)s_2(A)\cdots s_n(A) \qquad (n = 1, 2, \cdots, \nu(A)). \tag{3.6}$$

If $\nu(A) = r(A)\,(\leqq \infty)$, *then equality will hold simultaneously in all the relations* (3.6) *if and only if the operator* A *is normal.*

PROOF. In accordance with Lemma I.4.1, we choose an orthonormal Schur system $\{\omega_j\}_1^{\nu(A)}$ such that

$$A\omega_j = a_{j1}\omega_1 + a_{j2}\omega_2 + \cdots + a_{jj}\omega_j \qquad (j = 1, 2, \cdots, \nu(A)),$$

where

$$a_{jj} = (A\omega_j, \omega_j) = \lambda_j(A) \qquad (j = 1, 2, \cdots, \nu(A)). \tag{3.7}$$

By Lemma 3.1,

$$\det\|(A\omega_j, A\omega_k)\|_1^n \leqq s_1^2(A)s_2^2(A)\cdots s_n^2(A) \qquad (n \leqq \nu(A)). \tag{3.8}$$

Since

$$(A\omega_j, A\omega_k) = \sum_{q=1}^{\min(j,k)} (A\omega_j, \omega_q)\,\overline{(A\omega_k, \omega_q)},$$

it follows that

$$\begin{aligned}\det\|(A\omega_j, A\omega_k)\|_1^n &= \det\|(A\omega_j, \omega_k)\|_1^n \det\|\overline{(A\omega_j, \omega_k)}\|_1^n \\ &= |\det\|(A\omega_j, \omega_k)\|_1^n|^2.\end{aligned}$$

Taking into account that

$$\det\|(A\omega_j, \omega_k)\|_1^n = \lambda_1(A)\lambda_2(A)\cdots\lambda_n(A),$$

we obtain

$$\det\|(A\omega_j, A\omega_k)\|_1^n = |\lambda_1(A)|^2|\lambda_2(A)|^2\cdots|\lambda_n(A)|^2. \tag{3.9}$$

Comparing (3.8) and (3.9), we arrive at the relations (3.6). Let us now consider the case in which $\nu(A) = r(A)$ and equality obtains in all the relations (3.6). Then

$$|\lambda_j(A)| = s_j(A) \qquad (j = 1, 2, \cdots, r(A)).$$

From (3.7) it follows that

$$|(A\omega_j, \omega_j)| = s_j(A) \qquad (j = 1, 2, \cdots, r(A)),$$

and hence by Lemma 3.2 the operator A is normal.

The lemma is proved.

REMARK 3.1. It is obvious that for any finite-dimensional operator

A equality holds for the last of the relations (3.6), i.e., for $n = \nu(A)$.

A. Horn [2] showed that the relations (3.6), under the hypothesis that in the last one of them the inequality is replaced by equality, precisely characterize the interdependence between the s-numbers and eigenvalues of a finite-dimensional operator. This means that for any complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ and any nonnegative numbers $s_1, s_2, \cdots, s_n$, satisfying the conditions

$$|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_n|; \quad s_1 \geqq s_2 \geqq \cdots \geqq s_n;$$

$$|\lambda_1\lambda_2 \cdots \lambda_k| \leqq s_1 s_2 \cdots s_k \qquad (k = 1, 2, \cdots, n-1),$$

$$|\lambda_1\lambda_2 \cdots \lambda_n| = s_1 s_2 \cdots s_n$$

one can find an operator A, acting in n-dimensional space, such that

$$\lambda_j(A) = \lambda_j \quad \text{and} \quad s_j(A) = s_j \qquad (j = 1, 2, \cdots, n).$$

2. LEMMA 3.4 (WEYL [2]; HARDY, LITTLEWOOD AND PÓLYA [1]). *Let $\Phi(x)$ $(-\infty \leqq x < \infty)$ be a convex function, vanishing for $x = -\infty$ $(\Phi(-\infty) = \lim_{x\to\infty}\Phi(-x) = 0)$, and let $\{a_j\}_1^\omega$ and $\{b_j\}_1^\omega$ $(\omega \leqq \infty)$ be nonincreasing sequences of real numbers such that*

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} b_j \qquad (k = 1, 2, \cdots, \omega).$$

Then

$$\sum_{j=1}^{k} \Phi(a_j) \leqq \sum_{j=1}^{k} \Phi(b_j) \qquad (k = 1, 2, \cdots, \omega). \tag{3.10}$$

(In particular, if $\omega = \infty$ then $\sum_{j=1}^\infty \Phi(a_j) \leqq \sum_{j=1}^\infty \Phi(b_j)$.)

If in addition the function $\Phi(x)$ is strictly convex, then the equality

$$\sum_{j=1}^{\omega} \Phi(a_j) = \sum_{j=1}^{\omega} \Phi(b_j) \qquad (< \infty) \tag{3.11}$$

will hold if and only if

$$a_j = b_j \qquad (j = 1, 2, \cdots, \omega).$$

PROOF. Let us denote by $\Phi'(x)$ $(-\infty < x < \infty)$ the left derivative of the convex function $\Phi(x)$, which, as is well known, exists everywhere and is a nonnegative nondecreasing function. We shall prove that the function $\Phi(x)$ admits the representation

$$\Phi(x) = \int_{-\infty}^{\infty} (x - u)_+ \, d\Phi'(u), \tag{3.12}$$

where $y_+ = \max(y, 0)$.

In fact,

$$(3.13)\qquad \begin{aligned}\int_{-N}^{\infty}(x-u)_+\,d\Phi'(u) &= \int_{-N}^{x}(x-u)\,d\Phi'(u)\\ &= \int_{-N}^{x}\Phi'(u)\,du - (x+N)\Phi'(-N) \qquad (\geqq 0)\end{aligned}$$

where N is any positive number. From the positivity of the left side of (3.13) it follows that

$$(3.14)\qquad \begin{aligned}(x+N)\Phi'(-N) &\leqq \int_{-N}^{x}\Phi'(u)\,du\\ &= \Phi(x) - \Phi(-N) \leqq \Phi(x) \qquad (x > -N),\end{aligned}$$

and hence

$$(3.15)\qquad \overline{\lim_{N\to\infty}}\, N\Phi'(-N) < \infty, \quad \lim_{N\to\infty}\Phi'(-N) = 0.$$

Since by hypothesis $\Phi(x)\to 0$ as $x\to -\infty$, we conclude from (3.14) and (3.15) that

$$\lim_{N\to\infty}(x+N)\Phi'(-N) = \lim_{N\to\infty} N\Phi'(-N) = 0.$$

To obtain the representation (3.12) of the function $\Phi(x)$, it remains to pass to the limit $N\to\infty$ in (3.13).

From the representation (3.12) follows

$$(3.16)\qquad \sum_{j=1}^{k}\Phi(a_j) = \int_{-\infty}^{\infty} A_k(x)\,d\Phi'(x),$$

where

$$A_k(x) = \sum_{j=1}^{k}(a_j - x)_+.$$

Similarly,

$$(3.16')\qquad \sum_{j=1}^{k}\Phi(b_j) = \int_{-\infty}^{\infty} B_k(x)\,d\Phi'(x),$$

where

$$B_k(x) = \sum_{j=1}^{k}(b_j - x)_+.$$

The functions $A_k(x)$ and $B_k(x)$ are connected by the relations

(3.17) $\qquad A_k(x) \leqq B_k(x) \qquad (-\infty < x < \infty; k = 1, 2, \cdots).$

Indeed, for all x satisfying one of the inequalities

$$x \leqq \min(a_k, b_k), \quad x \geqq b_1,$$

the relation (3.17) is obvious. Now let

$$a_{q+1} \leqq x < a_q \quad \text{and} \quad b_{p+1} \leqq x < b_p \qquad (p, q \leqq k);$$

then for $p \geqq q$

$$A_k(x) = \sum_{j=1}^{q} a_j - qx \leqq \sum_{j=1}^{q} b_j - qx + (b_{q+1} - x) + \cdots + (b_p - x) = B_k(x),$$

and for $p < q$

$$A_k(x) = \sum_{j=1}^{q} a_j - qx \leqq \sum_{j=1}^{q} a_j - qx - (b_{p+1} - x) - \cdots - (b_q - x)$$

$$\leqq \sum_{j=1}^{p} b_j - px = B_k(x).$$

The validity of the inequalities (3.10) follows at once from the inequalities (3.17) and the representations (3.16) and (3.16′).

To clarify when equality holds in (3.11), we shall for definiteness consider the more difficult case $\omega = \infty$. In this case it follows from (3.17) that

$$A(x) = \sum_{j=1}^{\infty} (a_j - x)_+ \leqq B(x) = \sum_{j=1}^{\infty} (b_j - x)_+ \qquad (-\infty < x < \infty).$$

It follows from (3.10) that whenever the series $\sum_j \Phi(b_j)$ converges, the series $\sum_j \Phi(a_j)$ also converges; moreover, by virtue of (3.16) and (3.16′)

$$\sum_{j=1}^{\infty} \Phi(a_j) = \int_{-\infty}^{\infty} A(u)\, d\Phi'(u) \leqq \int_{-\infty}^{\infty} B(u)\, d\Phi'(u) = \sum_{j=1}^{\infty} \Phi(b_j).$$

Equality will hold in the last relation if and only if $A(x) = B(x)$ $(-\infty < x < \infty)$, which in turn is possible only if $a_j = b_j$ $(j = 1, 2, \cdots)$.

3. The above results enable us to establish a number of important results concerning the s-numbers of completely continuous operators.

THEOREM 3.1 (WEYL'S MAJORANT THEOREM; CF. WEYL [2]). *Let A be a completely continuous operator, and let $f(x)$ $(0 \leqq x < \infty;\ f(0) = 0)$ be a function which becomes convex following the substitution $x = e^t$*

$(-\infty < t < \infty)$. *Then*

$$\sum_{j=1}^{k} f(|\lambda_j|) \leqq \sum_{j=1}^{k} f(s_j) \qquad (k = 1, 2, \cdots, \nu(A)), \tag{3.18}$$

where

$$\lambda_j = \lambda_j(A),\ s_j = s_j(A) \qquad (j = 1, 2, \cdots, \nu(A)).$$

In particular, if $\nu(A) = \infty$,

$$\sum_{j=1}^{\infty} f(|\lambda_j|) \leqq \sum_{j=1}^{\infty} f(s_j). \tag{3.19}$$

If the function $\Phi(t) = f(e^t)$ *is strictly convex, then the equality*

$$\sum_{j=1}^{\nu(A)} f(|\lambda_j|) = \sum_{j=1}^{\infty} f(s_j) \qquad (< \infty),^{4)} \tag{3.20}$$

under the hypothesis that the right side is finite, holds if and only if the operator A *is normal.*

PROOF. This theorem is a simple combination of the functional-analytic Lemma 3.3 and the purely function-theoretic Lemma 3.4. In fact, by virtue of Lemma 3.3, Lemma 3.4 is applicable to the numbers

$$a_j = \ln|\lambda_j|, \quad b_j = \ln s_j \qquad (j = 1, 2, \cdots, \nu(A))$$

and the function $\Phi(t) = f(e^t)$. From this one obtains the relations (3.18) and (3.19).

Let us now consider the case in which the function $\Phi(t)$ is strictly convex and (3.20) is fulfilled. If $\nu(A)$ is finite, then, setting $k = \nu(A)$ in (3.18) and comparing with (3.20), we obtain

$$\sum_{j=1}^{\nu(A)} f(|\lambda_j|) = \sum_{j=1}^{\nu(A)} f(s_j);\ s_j(A) = 0 \quad \text{for} \quad j > \nu(A).$$

Here we have taken into account that $f(x) > 0$ for $x > 0$. Thus $r(A) = \nu(A)$ for the case being considered, and moreover, by Lemma 3.4,

$$|\lambda_j(A)| = s_j(A) \qquad (j = 1, 2, \cdots, r(A)). \tag{3.21}$$

---

[4] We recall that even for a finite-dimensional operator A the s-numbers form an infinite sequence (see §2.1). We also recall that the eigenvalues λ_j are numbered according to decreasing modulus, taking into account their algebraic multiplicity, and therefore the upper limit $\nu(A)$ in the sum $\sum_{j=1}^{\nu(A)}$ indicates that every number λ_j enters into this sum as many times as its multiplicity.

By the same Lemma 3.4 for $\nu(A) = \infty$, the equality (3.21) will follow for all $j = 1, 2, \cdots$ from the equality (3.20).

Recalling Lemma 3.3, we conclude that in each of the two cases ($\nu(A) < \infty$; $\nu(A) = \infty$) the equality (3.21) implies the normality of the operator A.

Since, conversely, for normal operators A we have $\nu(A) = r(A)$, and the equalities (3.21), along with the equality (3.20), are valid, the theorem is proved.

COROLLARY 3.1. *For any $A \in \mathfrak{S}_\infty$ one has the relations*

$$(3.22) \qquad \sum_{j=1}^{n} |\lambda_j(A)|^p \leqq \sum_{j=1}^{n} s_j^p(A) \qquad (p > 0; n = 1, 2, \cdots, \nu(A)),$$

and also the relations

$$(3.23) \qquad \prod_{j=1}^{n} (1 + r|\lambda_j(A)|) \leqq \prod_{j=1}^{n} (1 + rs_j(A)) \qquad (n = 1, 2, \cdots, \nu(A)),$$

where r is any positive number.

In fact, (3.22) and (3.23) are the special cases of the relations (3.18) obtained, respectively, for $f(x) = x^p$ and $f(x) = \ln(1 + rx)$. It is easily verified that these functions become convex following the substitution $x = e^t$.

COROLLARY 3.2. *Let $A \in \mathfrak{S}_\infty$ and suppose that for some $\rho > 0$*

$$(3.24) \qquad s_n(A) = O(n^{\;1/\rho}) \qquad (n \to \infty)$$

or

$$(3.25) \qquad s_n(A) = o(n^{-1/\rho}) \qquad (n \to \infty).$$

Then if the operator A has an infinite number of eigenvalues, one has in the case of (3.24)

$$(3.26) \qquad \lambda_n(A) = O(n^{-1/\rho}) \qquad (n \to \infty),$$

and in the case of (3.25)

$$(3.27) \qquad \lambda_n(A) = o(n^{\;1/\rho}) \qquad (n \to \infty).$$

In fact, let us first assume that $\rho < 1$; then the entire function

$$f_s(z) = \prod_{n=1}^{\infty} (1 + s_n(A) z),$$

according to well-known theorems of the theory of functions (cf. B. Ja. Levin [1]), will be an entire function of order ρ and moreover of normal type in the case of (3.24):

$$\ln f_s(r) = O(r^\rho) \qquad (r \uparrow \infty)$$

and of minimal type in the case of (3.25):

$$\ln f_s(r) = o(r^\rho) \qquad (r \uparrow \infty).$$

But then the same assertions are valid for the function

$$f_\lambda(z) = \prod_{j=1}^{\infty}(1 + \lambda_j(A)z),$$

since, by (3.23),

$$\max_{|z| \leq r} | f_\lambda(z) | \leqq f_s(r).$$

Therefore, according to the inverse theorems of the theory of entire functions concerning the connection between the order of their growth and the order of growth of their zeros, we will have, respectively, (3.26) and (3.27).

Let us now consider the case in which $\rho > 1$.

Choosing an integer $\nu > \rho$, we form the numbers $\lambda_n^\nu(A)$ and $s_n^\nu(A)$, for which we shall again have

$$\left| \prod_{j=1}^{\infty}(1 + \lambda_j^\nu(A)z) \right| \leq \prod_{j=1}^{\infty}(1 + s_j^\nu(A)|z|).$$

Hence we can again conclude that the numbers $\lambda_n^\nu(A)$ will have the same order of decrease as the numbers $s_n^\nu(A)$. The assertion is proved.

REMARK 3.2. Corollary 3.2 remains valid if, in its statement, $n^{-1/\rho}$ is replaced by $n^{-1/\rho}L(n)$, where $L(r)$ $(0 < r < \infty)$ is a slowly varying function (see, concerning such functions, Chapter III, §14.4). Here, instead of theorems on entire functions of order ρ, it will be necessary to use theorems of the theory of functions concerning the proximate order of growth (cf. B. Ja. Levin [1], Chapter I, §13).

4. In Theorem 3.1 the question is one of inequalities of the form

$$\Phi(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|) \leqq \Phi(s_1, s_2, \cdots, s_n),$$

where the function Φ has an additive structure, namely:

$$\Phi(t_1, t_2, \cdots, t_n) = \sum_{j=1}^{n} \Phi(t_j). \tag{3.28}$$

As A. Ostrowski [1] showed (for the case of finite-dimensional operators), the results of Weyl admit generalization to the case of a wider class of functions Φ, not subject to the restriction (3.28).

LEMMA 3.5.[5] *Let the function* $\Phi(t_1, t_2, \cdots, t_n)$ *be defined in the region* D_n,

$$-\infty < t_n \leqq t_{n-1} \leqq \cdots \leqq t_2 \leqq t_1 < \infty,$$

and have there continuous partial derivatives, satisfying the conditions

$$\partial\Phi/\partial t_1 > \partial\Phi/\partial t_2 > \cdots > \partial\Phi/\partial t_n > 0 \quad \text{for} \quad t_1 > t_2 > \cdots > t_n. \tag{3.29}$$

Then for any two nonincreasing sequences of real numbers $\{a_j\}_1^n$ *and* $\{b_j\}_1^n$ *satisfying the conditions*

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} b_j \qquad (k = 1, 2, \cdots, n), \tag{3.30}$$

the inequality

$$\Phi(a_1, a_2, \cdots, a_n) \leqq \Phi(b_1, b_2, \cdots, b_n) \tag{3.31}$$

holds; the equal sign holds only for $a_j = b_j$ $(j = 1, 2, \cdots, n)$.

PROOF. Let us consider the transformation

$$s_k = \sum_{j=1}^{k} t_j \qquad (k = 1, 2, \cdots, n), \tag{3.32}$$

which maps the region D_n into the region Ω_n of all points $(s_1, s_2, \cdots, s_n)$ for which

$$s_k - s_{k-1} \geqq s_{k+1} - s_k \qquad (k = 1, 2, \cdots, n-1; s_0 = 0).$$

Under this transformation the function Φ goes over into some function Ψ:

$$\Psi(s_1, s_2, \cdots, s_n) = \Phi(t_1, t_2, \cdots, t_n),$$

defined in Ω_n and satisfying there the conditions

$$\begin{aligned} &\partial\Psi/\partial s_k = \partial\Phi/\partial t_k - \partial\Phi/\partial t_{k+1} > 0 \qquad (k = 1, 2, \cdots, n-1), \\ &\partial\Psi/\partial s_n = \partial\Phi/\partial t_n > 0. \end{aligned} \tag{3.33}$$

Under the mapping (3.32) the segment

$$t_k = (1-\lambda)a_k + \lambda b_k \qquad (0 \leqq \lambda \leqq 1;\ k = 1, 2, \cdots, n)$$

[5] This lemma represents a simple modification of a result of A. Ostrowski [1]. Lemma 3.5, Remark 3.3 and the possibility of deriving Theorem 3.2 from this lemma were brought to our attention by Ju. A. Palant.

goes over into the segment S:

$$s_k = (1 - \lambda)\alpha_k + \lambda\beta_k \qquad (0 \leqq \lambda \leqq 1;\ k = 1, 2, \cdots, n),$$

where by hypothesis

$$\alpha_k = \sum_{j=1}^{k} a_j \leqq \beta_k = \sum_{j=1}^{k} b_j \qquad (k = 1, 2, \cdots, n).$$

If the segment S does not degenerate to a point, i.e. if $\alpha_k < \beta_k$ for at least one k $(= 1, 2, \cdots, n)$, then the function Ψ is an increasing function of the argument λ on S since, according to (3.33),

$$\frac{\partial \Psi}{\partial \lambda} = \sum_{j=1}^{n} \frac{\partial \Psi}{\partial s_j}(\beta_j - \alpha_j) > 0.$$

Hence

$$\Psi|_{\lambda=1} = \Phi(b_1, b_2, \cdots, b_n) > \Psi|_{\lambda=0} = \Phi(a_1, a_2, \cdots, a_n),$$

as was to be proved.

REMARK 3.3. It is easily seen that inequality (3.31) will also be fulfilled in the case where the conditions (3.29) are replaced by the weakened conditions

$$\frac{\partial \Phi}{\partial t_1} \geqq \frac{\partial \Phi}{\partial t_2} \geqq \cdots \geqq \frac{\partial \Phi}{\partial t_n} \geqq 0. \tag{3.34}$$

On the other hand, if the convex function $\Phi(t)$ $(-\infty \leqq t < \infty)$ is continuously differentiable, then the function $\Phi(t_1, t_2, \cdots, t_n)$ of the form (3.28) will satisfy the conditions (3.34), and also the conditions (3.29) if $\Phi(t)$ is strictly convex.

Thus, for continuously differentiable functions $\Phi(t)$, Lemma 3.4 for finite sequences $\{a_j\}_1^n$ and $\{b_j\}_1^n$ is a corollary of Lemma 3.5.

As a corollary of Lemmas 3.3 and 3.5, we obtain a theorem which is due basically to A. Ostrowski [1].

THEOREM 3.2. *Let A be a completely continuous operator, and let $F(x_1, x_2, \cdots, x_n)$ $(0 < x_n \leqq x_{n-1} \leqq \cdots \leqq x_1 < \infty;\ n \leqq \nu(A))$ be a function which has continuous partial derivatives, satisfying the conditions*:

I) $$\partial F/\partial x_k > 0 \qquad (k = 1, 2, \cdots, n);$$

II) $$x_{k+1}\partial F/\partial x_{k+1} < x_k \partial F/\partial x_k \quad \textit{for} \quad x_{k+1} < x_k \qquad (k = 1, 2, \cdots, n-1).$$

Then

$$F(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|) \leqq F(s_1, s_2, \cdots, s_n), \tag{3.35}$$

where $\lambda_j = \lambda_j(A)$, $s_j = s_j(A)$ $(j = 1, 2, \cdots, n)$, *and equality holds if and only if* $|\lambda_j| = s_j$ $(j = 1, 2, \cdots, n)$.

PROOF. In fact, if the function F satisfies the assumptions of the theorem, then the function

$$\Phi(t_1, t_2, \cdots, t_n) = F(e^{t_1}, e^{t_2}, \cdots, e^{t_n}) \qquad (-\infty < t_n \leqq t_{n-1} \leqq \cdots \leqq t_1 < \infty)$$

satisfies the assumptions of Lemma 3.5.

On the other hand, by Lemma 3.3, the numbers $a_j = \ln|\lambda_j|$, $b_j = \ln s_j$ $(j = 1, 2, \cdots, n)$ satisfy the conditions (3.30), and therefore (3.31) holds for them, which by the indicated choice of the function Φ yields the relation (3.35).

The theorem is proved.

It is not hard to see that the conditions I) and II) of Theorem 3.2 are satisfied for the elementary symmetric functions

$$\Sigma_n(x_1, x_2, \cdots, x_n) = \sum_{1 \leqq j_1 < j_2 < \cdots < j_n \leqq k} x_{j_1} x_{j_2} \cdots x_{j_n},$$

and thus one has

COROLLARY 3.3. *For any completely continuous operator* A

$$(3.36) \qquad \Sigma_n(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_k|) \leqq \Sigma_n(s_1, s_2, \cdots, s_k) \qquad (1 \leqq n \leqq k \leqq \nu(A)),$$

where $\lambda_j = \lambda_j(A)$, $s_j = s_j(A)$ $(j = 1, 2, \cdots, k)$.

REMARK 3.4. For $n = k$ the inequality (3.36) says the same thing as does the inequality (3.6). H. Weyl obtained the latter inequality by an argument different from that presented here; Weyl's argument is interesting in that it leads to many new inequalities, not included within the scheme of Ostrowski's theorem (Theorem 3.2).

We present Weyl's argument. To do this we form, given a positive integer k, all possible products of the form

$$\Lambda^{(k)}(A) = \lambda_{j_1}(A)\lambda_{j_2}(A) \cdots \lambda_{j_k}(A) \qquad (j_1 < j_2 < \cdots < j_k)$$

and arbitrarily enumerate them in order of decreasing modulus. Denote the numbers thus obtained by $\Lambda_1^{(k)}, \Lambda_2^{(k)}, \cdots$. Obviously

$$\Lambda_1^{(k)} = \lambda_1(A)\lambda_2(A) \cdots \lambda_k(A).$$

Proceeding similarly with the numbers $s_j(A)$, we construct a sequence of numbers $S_j^{(k)}$ $(j = 1, 2, \cdots)$; in particular,

$$S_1^{(k)} = s_1(A)s_2(A) \cdots s_k(A).$$

Just as in the matrix case (H. Weyl, by the way, carried out his argument for matrices), the numbers

$$\Lambda_j^{(k)} \quad (j = 1, 2, \cdots, N;\ N = C^k_{\nu(A)}) \quad \text{and} \quad S_j^{(k)} \quad (j = 1, 2, \cdots)$$

are, respectively, the eigenvalues and s-numbers of a certain operator $\mathfrak{A} = A^{(k)}$, called the *kth associated operator* with respect to A, and acting in some new Hilbert space $\mathfrak{H}^{(k)}$ (see F. R. Gantmaher and M. G. Kreĭn [1]).

If $A\phi_1 = \lambda_1(A)\phi_1$, then

$$|\lambda_1(A)| = \frac{|A\phi_1|}{|\phi_1|} \leqq \max_{\phi \in \mathfrak{H}} \frac{|A\phi|}{|\phi|} = s_1(A).$$

Thus the inequality $|\lambda_1(A)| \leqq s_1(A)$ is established at once. On the other hand, applying this inequality to the operator $\mathfrak{A} = A^{(k)}$ yields $|\Lambda_1^{(k)}| \leqq S_1^{(k)}$, which is equivalent to (3.6).

It is interesting to note that Theorem 3.1, which was obtained on the basis of the last inequality, upon application to the operator $\mathfrak{A}$ enables one to assert that for every function $f(x)$, satisfying the hypothesis of Theorem 3.1, one has for any positive integer k $(\leqq \nu(A))$

$$\sum_{j=1}^{\kappa} f(|\Lambda_j^{(k)}|) \leqq \sum_{j=1}^{\kappa} f(S_j^{(k)}) \qquad (\kappa = 1, 2, \cdots).$$

In particular, for $f(x) = x$ we obtain

$$\sum_{j=1}^{\kappa} |\Lambda_j^{(k)}| \leqq \sum_{j=1}^{\kappa} S_j^{(k)} \qquad (\kappa = 1, 2, \cdots). \tag{3.37}$$

Let us emphasize that the inequalities (3.37), generally speaking, are of a different character than the inequalities (3.36), although in certain cases for various values of $n \leqq k$ the inequality (3.37) can assert the same thing as inequality (3.36) with the same k and with κ equal to C_k^n, the number of combinations of k objects taken n at a time.

The relations (3.37) in turn enable one to assert on the basis of Theorem 3.2 that inequalities of the type (3.36) hold for the numbers $\Lambda_j^{(k)}$ and $S_j^{(k)}$ $(j = 1, 2, \cdots)$.

§4. Inequalities for the s-numbers of sums and products of completely continuous operators

In this section we discuss, with a number of additions, results of K. Fan [3] and A. Horn [1] which are closely related to Weyl's investigations.

1. The following simple lemma is extremely useful.

LEMMA 4.1 (K. FAN [3]). *Let $A \in \mathfrak{S}_\infty$. Then for any positive integer n*

$$\max\left|\sum_{j=1}^{n}(UA\phi_j,\phi_j)\right| = \sum_{j=1}^{n} s_j(A), \tag{4.1}$$

where the maximum is taken over all unitary operators U and orthonormal systems $\{\phi_j\}_1^n$. In particular,

$$\sum_{j=1}^{n}|(A\phi_j,\phi_j)| \leqq \sum_{j=1}^{n} s_j(A). \tag{4.2}$$

PROOF. Let us denote by P the orthoprojector onto the subspace with basis ϕ_j $(j = 1, 2, \cdots, n)$. Then

$$\sum_{j=1}^{n}(UA\phi_j,\phi_j) = \sum_{j=1}^{n}(A_1\phi_j,\phi_j),$$

where $A_1 = PUAP$. By a well-known result of linear algebra

$$\mathbf{sp}\, A_1 = \sum_{j=1}^{n}(A_1\phi_j,\phi_j) = \sum_{j=1}^{n}\lambda_j(A_1).$$

By virtue of Corollary 3.1

$$\sum_{j=1}^{n}|\lambda_j(A_1)| \leqq \sum_{j=1}^{n} s_j(A_1).$$

On the other hand, according to the bounds (2.2) and (2.3)

$$s_j(A_1) = s_j(PUAP) \leqq s_j(A) \qquad (j = 1, 2, \cdots, n).$$

Therefore

$$\left|\sum_{j=1}^{n}(UA\phi_j,\phi_j)\right| \leqq \sum_{j=1}^{n} s_j(A).$$

We shall show that the equality is attained here. To do this we represent the operator A in polar form $A = VH$ and denote by e_j $(j = 1, 2, \cdots)$ a complete orthonormal system of eigenvectors of the operator H.

Obviously there exists a unitary operator U such that

$$UAe_j = V^*Ae_j \ (= He_j) \qquad (j = 1, 2, \cdots, n).$$

For this operator we have

$$\sum_{j=1}^{n}(UAe_j,e_j) = \sum_{j=1}^{n}(He_j,e_j) = \sum_{j=1}^{n} s_j(A).$$

Thus the relation (4.1) is established. The relation (4.2) follows at once;

indeed, if U_0 is any unitary operator such that

$$U_0^*\phi_j = e^{i\theta_j}\phi_j,\ \theta_j = \arg(A\phi_j, \phi_j) \qquad (j = 1, 2, \cdots, n),$$

then

$$\sum_{j=1}^{n} |(A\phi_j, \phi_j)| = \sum_{j=1}^{n} (U_0 A\phi_j, \phi_j) \leqq \sum_{j=1}^{n} s_j(A).$$

Lemma 4.1 is proved.

The following lemma is now easily established.

LEMMA 4.2. (A. HORN [1], K. FAN [3]). *For any operators $A, B \in \mathfrak{S}_\infty$ one has the relations*

$$\prod_{j=1}^{n} s_j(AB) \leqq \prod_{j=1}^{n} s_j(A) \prod_{j=1}^{n} s_j(B) \qquad (n = 1, 2, \cdots), \tag{4.3}$$

$$\sum_{j=1}^{n} s_j(A + B) \leqq \sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B) \qquad (n = 1, 2, \cdots). \tag{4.4}$$

PROOF. For any complete orthonormal system of vectors e_j ($j = 1, 2, \cdots$) it follows from Lemma 3.1 that

$$\begin{aligned} \det \| (ABe_j, ABe_k) \|_1^n &\leqq \prod_{j=1}^{n} s_j^2(A) \cdot \det \| (Be_j, Be_k) \|_1^n \\ &\leqq \prod_{j=1}^{n} s_j^2(A) \prod_{j=1}^{n} s_j^2(B). \end{aligned} \tag{4.5}$$

Choosing for the e_j a complete system of eigenvectors of the operator B^*A^*AB, we will have

$$\det \| (ABe_j, ABe_k) \|_1^n = \prod_{j=1}^{n} s_j^2(AB);$$

then (4.3) follows from (4.5).

By Lemma 4.1 we can choose an orthonormal system of vectors ϕ_j ($j = 1, 2, \cdots, n$) and a unitary operator U such that

$$\left| \sum_{j=1}^{n} (U(A + B)\phi_j, \phi_j) \right| = \sum_{j=1}^{n} s_j(A + B). \tag{4.6}$$

Hence, on the basis of the same Lemma 4.1, we obtain

$$\sum_{j=1}^{n} s_j(A + B) \leqq \left| \sum_{j=1}^{n} (UA\phi_j, \phi_j) | + | \sum_{j=1}^{n} (UB\phi_j, \phi_j) \right| \leqq \sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B).$$

Lemma 4.2 is proved.

The inequality (4.3) was pointed out by A. Horn [1], and the inequality (4.4) by K. Fan [3].

2. With the help of the lemmas just proved, we can immediately establish the following results.

THEOREM 4.1 (K. FAN [3]). *Let A and B be completely continuous linear operators, and $f(x)$ $(0 \leqq x < \infty)$ a nondecreasing convex function which vanishes for $x = 0$. Then*

$$\sum_{j=1}^{k} f(s_j(A+B)) \leqq \sum_{j=1}^{k} f(s_j(A) + s_j(B)) \qquad (k = 1, 2, \cdots), \tag{4.7}$$

and consequently

$$\sum_{j=1}^{\infty} f(s_j(A+B)) \leqq \sum_{j=1}^{\infty} f(s_j(A) + s_j(B)). \tag{4.8}$$

PROOF. Indeed, the relations (4.7) and (4.8) follow directly from Lemmas 4.2 and 3.4, if we put in Lemma 3.4

$$a_j = s_j(A+B),$$
$$b_j = s_j(A) + s_j(B) \quad \text{and} \quad \Phi(x) = \begin{cases} f(x) & (0 \leqq x < \infty) \\ 0 & (-\infty \leqq x < 0). \end{cases}$$

THEOREM 4.2 (A. HORN [1]). *If the function $f(x)$ $(0 \leqq x < \infty;\ f(0) = 0)$ becomes convex following the substitution $x = e^t$ $(-\infty \leqq t < \infty)$, then for any linear completely continuous operators A and B*

$$\sum_{j=1}^{k} f(s_j(AB)) \leqq \sum_{j=1}^{k} f(s_j(A)s_j(B)) \qquad (k = 1, 2, \cdots), \tag{4.9}$$

and consequently

$$\sum_{j=1}^{\infty} f(s_j(AB)) \leqq \sum_{j=1}^{\infty} f(s_j(A)s_j(B)). \tag{4.10}$$

PROOF. In fact, by virtue of Lemmas 4.2 and 3.4 the relations (4.9) and (4.10) follow at once from the relations (3.10) and (3.11), if in the latter we put

$$a_j = \ln s_j(AB) \quad \text{and} \quad b_j = \ln[s_j(A)s_j(B)].$$

COROLLARY 4.1. *For any linear completely continuous operators A and B,*

$$\sum_{j=1}^{k} s_j(AB) \leqq \sum_{j=1}^{k} s_j(A)s_j(B) \qquad (k = 1, 2, \cdots). \tag{4.11}$$

In fact, the relations (4.11) are obtained from (4.9) for $f(t) = t$.

The inequalities (4.7)—(4.11) can naturally be generalized to the case of n operators $A_1, A_2, \cdots, A_n$.

COROLLARY 4.2. *For any linear completely continuous operator A and any numbers $n = 1, 2, \cdots; p > 0$ one has the relations*

$$\sum_{j=1}^{k} s_j^{p/n}(A^n) \leqq \sum_{j=1}^{n} s_j^p(A) \qquad (k = 1, 2, \cdots). \tag{4.12}$$

The relations (4.12) follow from (4.9), written for n operators $A_j = A$ $(j = 1, 2, \cdots, n)$ and for the function $f(t) = t^{p/n}$.

3. If $A \in \mathfrak{S}_\infty$, and $\{\phi_j\}_1^\omega$ $(\omega \leqq \infty)$ is some orthonormal system, then according to (4.2)

$$\sum_{j=1}^{k} |(A\phi_j, \phi_j)| \leqq \sum_{j=1}^{k} s_j(A) \qquad (k = 1, 2, \cdots, \omega).$$

On the basis of Lemma 3.4 one can assert, for any nondecreasing convex function $f(x)$ $(0 \leqq x < \infty; f(0) = 0)$, that

$$\sum_{j=1}^{k} f(t_j) \leqq \sum_{j=1}^{k} f(s_j) \qquad (k = 1, 2, \cdots, \omega), \tag{4.13}$$

where

$$t_j = |(A\phi_j, \phi_j)|, \; s_j = s_j(A) \qquad (j = 1, 2, \cdots, \omega).$$

In particular, if $\omega = \infty$,

$$\sum_{j=1}^{\infty} f(t_j) \leqq \sum_{j=1}^{\infty} f(s_j). \tag{4.14}$$

We shall show that if the function $f(x)$ is strictly convex and the right side of the relation (4.14) is finite, then equality will hold if and only if

$$A = \sum_{j=1}^{\omega} (A\phi_j, \phi_j)(\cdot, \phi_j)\phi_j. \tag{4.15}$$

Without loss of generality we can assume that $t_j > 0$ for all j and that $t_1 \geqq t_2 \geqq \cdots$; otherwise we could renumber the system $\{\phi_j\}_1^\omega$ after first discarding from it those ϕ_j for which $(A\phi_j, \phi_j) = 0$. Accepting this, we can assert on the basis of Lemma 3.4 that equality will hold in (4.14) if and only if

$$|(A\phi_j, \phi_j)| = s_j(A) \qquad (j = 1, 2, \cdots, \omega).$$

On the basis of Lemma 3.2 we conclude that the operator A is normal,

and that $\{\phi_j\}_1^\omega$ is a system of its eigenvectors, complete in $\overline{\mathfrak{R}(A)}$, which is equivalent to (4.15).

§5. Some generalizations of the preceding inequalities

The relations (4.13) and (4.14) admit a number of generalizations. Before formulating them, we shall make some remarks.

Let $\{P_k\}_1^\omega$ $(\omega \leqq \infty)$ be some system of mutually orthogonal orthoprojectors, so that

$$P_j P_k = 0 \qquad (j \neq k;\ j,k = 1,2,\cdots,\omega).$$

Then with every $A \in \mathfrak{R}$ one can associate the "diagonal-cell" operator

$$\hat{A} = \sum_{j=1}^{\omega} P_j A P_j, \tag{5.1}$$

which again belongs to $\mathfrak{R}$. Only the case $\omega = \infty$ requires clarification. In this case the equality (5.1) is to be understood in the sense of the strong convergence of operators, i.e. in the sense that for any $f \in \mathfrak{H}$

$$\hat{A}f = \sum_{j=1}^{\infty} P_j A P_j f.$$

The convergence, for any $f \in \mathfrak{H}$, of the series at the right follows from the fact that the elements $g_k = P_k A P_k f$ form an orthogonal system and

$$|g_k| \leqq |A|\,|P_k f|,$$

so that

$$\left|\sum_{j=m}^{n} g_j\right|^2 = \sum_{j=m}^{n} |g_j|^2 \leqq |A|^2 \sum_{j=m}^{n} |P_j f|^2.$$

If A is a completely continuous operator, then $\hat{A}$ will be one also, and when $\omega = \infty$ the series (5.1) will then converge in the sense of convergence in the uniform norm.

In fact, if in this case we introduce the lineals

$$\mathfrak{L}_n = \mathfrak{R}(P_1 + \cdots + P_n) \qquad (n = 1,2,\cdots),$$

then for any $\epsilon > 0$ there exists (by Theorem III.6.3) $N = N_\epsilon$ such that

$$|Af| \leqq \epsilon |f| \quad \text{for} \quad f \in \mathfrak{L}_N^\perp;$$

therefore

$$|P_k A P_k f| \leqq \epsilon |P_k f| \quad \text{for} \quad k > N.$$

Hence for $m, n > N$

$$\left|\sum_{j=m}^{n} P_j A P_j f\right| = \left(\sum_{j=m}^{n} |g_j|^2\right)^{1/2} \leqq \epsilon\left(\sum_{j=m}^{n} |P_j f|^2\right)^{1/2} \leqq \epsilon |f|,$$

which proves the uniform convergence of the series (5.1) and, at the same time, the complete continuity of the operator $\hat{A}$.

Let us note that for the dimensions $r(A)$ and $r(\hat{A})$ of the operators A and $\hat{A}$, all of the following three cases are possible:

$$1)\ r(\hat{A}) = r(A)\ (\leqq \infty), \qquad 2)\ r(\hat{A}) < r(A)\ (\leqq \infty),$$

$$3)\ r(A) < r(\hat{A})\ (\leqq \infty).$$

It is easy, for example, to construct operators for which $r(A) = 1$ and $r(\hat{A}) = \infty$.

THEOREM 5.1. *Let A be an operator from $\mathfrak{S}_\infty$, $\{P_k\}_1^\omega$ ($\omega \leqq \infty$) a system of mutually orthogonal orthoprojectors, and $\hat{A}$ the operator defined by* (5.1). *Then*

$$\sum_{j=1}^{n} s_j(\hat{A}) \leqq \sum_{j=1}^{n} s_j(A) \qquad (n = 1, 2, \cdots). \tag{5.2}$$

In order that

$$s_j(\hat{A}) = s_j(A) \qquad (j = 1, 2, \cdots), \tag{5.3}$$

it is necessary and sufficient that $A = \hat{A}$.

PROOF. Let us put

$$A_k = P_k A P_k, \quad r_k = r(A_k) \qquad (k = 1, 2, \cdots, \omega).$$

By Lemma 4.1, for every $k = 1, 2, \cdots, \omega$ and positive integer $n_k \leqq r_k$ we can find a unitary operator U_k, acting in the subspace $\mathfrak{R}(P_k)$ ($k = 1, 2, \cdots, \omega$), and a system of unit vectors $\phi_j^{(k)}$ ($j = 1, 2, \cdots, n_k$) such that

$$\sum_{j=1}^{n_k} (U_k A_k \phi_j^{(k)}, \phi_j^{(k)}) = \sum_{j=1}^{n_k} s_j(A_k).$$

Let U be the unitary operator defined on the entire space $\mathfrak{H}$ by

$$Uf = \sum_{k=1}^{\omega} U_k P_k f + Qf \qquad (f \in \mathfrak{H}),$$

where $Q = I - \sum_{k=1}^{\omega} P_k$. Then, as is easily seen,

$$(U_k A_k \phi_j^{(k)}, \phi_j^{(k)}) = (UA\phi_j^{(k)}, \phi_j^{(k)}) \qquad (j = 1, 2, \cdots, n_k;\ k = 1, 2, \cdots, \omega),$$

and consequently

$$\sum_{k=1}^{m}\sum_{j=1}^{n_k} s_j(A_k) = \sum_{k=1}^{m}\sum_{j=1}^{n_k} (UA\phi_j^{(k)}, \phi_j^{(k)}) \qquad (m = 1, 2, \cdots).$$

Hence we conclude from Lemma 4.1 that

$$\sum_{k=1}^{m}\sum_{j=1}^{n_k} s_j(A_k) \leqq \sum_{j=1}^{N} s_j(A). \tag{5.4}$$

Let us emphasize that in this relation m is any positive integer $\leqq \omega$, n_k is an arbitrary positive integer $\leqq r_k$ $(k = 1, 2, \cdots, m)$, and $N = \sum_{j=1}^{m} n_j$.

Noting that the sequence $\{s_j(\hat{A})\}_1^{r(\hat{A})}$ is just the collection of numbers

$$s_j(A_k) \qquad (j = 1, 2, \cdots, r_k;\ k = 1, 2, \cdots, \omega),$$

enumerated in decreasing order, we see that the inequalities (5.2) follow from the inequalities (5.4) (and conversely).

We shall now show that $\hat{A} = A$ follows from the equalities (5.3). We first consider the case in which A is positive.

In this case $\hat{A}$ is positive, and we can obviously find an orthonormal system $\phi_1, \phi_2, \cdots$ of eigenvectors of $\hat{A}$,

$$\hat{A}\phi_j = s_j(\hat{A})\phi_j \qquad (j = 1, 2, \cdots),$$

which is complete in $\Re(\hat{A})$ and such that each ϕ_j lies in the range of some P_k. We have

$$s_1(\hat{A}) = (\hat{A}\phi_1, \phi_1) = \sum_{k=1}^{\omega} (P_k A P_k \phi_1, \phi_1) = (A\phi_1, \phi_1).$$

Since by hypothesis $s_1(A) = s_1(\hat{A})$, it follows from the theorem on the minimax properties of eigenvalues that ϕ_1 is an eigenvector of A corresponding to the eigenvalue $s_1(A)$. Applying the same argument to the operators

$$\hat{A}_1 = \hat{A} - s_1(A)(\cdot, \phi_1)\phi_1, \qquad A_1 = A - s_1(A)(\cdot, \phi_1)\phi_1,$$

$$\hat{A}_1 = \sum_{k=1}^{\omega} P_k A_1 P_k,$$

and continuing this process, we find that $\phi_1, \phi_2, \cdots$ are eigenvectors of the operator A corresponding to the eigenvalues $s_1(A), s_2(A), \cdots$ from which it follows at once that $\hat{A} = A$.

To prove the same result for an arbitrary A, we put

$$C = A^*A, \qquad \hat{C} = \sum_{k=1}^{\omega} P_k C P_k.$$

Since

$$P_kA^*P_kAP_k \leqq P_kA^*AP_k \qquad (k = 1, 2, \cdots, \omega), \tag{5.5}$$

it follows that

$$\hat{A}^*\hat{A} = \sum_{k=1}^{\omega} P_kA^*P_kAP_k \leqq \sum_{k=1}^{\omega} P_kA^*AP_k = \hat{C}, \tag{5.6}$$

and consequently that

$$s_j^2(\hat{A}) = \lambda_j(\hat{A}^*\hat{A}) \leqq \lambda_j(\hat{C}) \qquad (j = 1, 2, \cdots).$$

On the other hand, applying the already proved inequalities (5.2) to C and $\hat{C}$ yields

$$\sum_{j=1}^{n} \lambda_j(\hat{C}) \leqq \sum_{j=1}^{n} \lambda_j(C) = \sum_{j=1}^{n} s_j^2(A) \qquad (n = 1, 2, \cdots),$$

and since by (5.3) and (5.6)

$$s_j^2(A) = s_j^2(\hat{A}) \leqq \lambda_j(\hat{C}) \qquad (j = 1, 2, \cdots),$$

we conclude that

$$\lambda_j(\hat{A}^*\hat{A}) = s_j^2(\hat{A}) = \lambda_j(\hat{C}) = \lambda_j(C) \qquad (j = 1, 2, \cdots).$$

Since C is a positive operator, it follows from what was proved earlier that $\hat{C} = C$. Recalling Lemma 1.1, we also see that $\hat{A}^*\hat{A} = \hat{C}$.

Therefore, according to (5.5) and (5.6),

$$P_kA^*P_kAP_k = P_kA^*AP_k \qquad (k = 1, 2, \cdots, \omega),$$

or, what is the same,

$$P_kA^*(I - P_k)AP_k = 0 \qquad (k = 1, 2, \cdots, \omega).$$

Since $I - P_k = (I - P_k)^2$, the last relation is equivalent to

$$(I - P_k)AP_k = 0 \qquad (k = 1, 2, \cdots, \omega).$$

Multiplying this relation by P_j $(j \neq k)$ or $Q = I - \sum_{k=1}^{\omega} P_k$ yields

$$QAP_k = 0, \quad P_jAP_k = 0 \qquad (j, k = 1, 2, \cdots, \omega;\ j \neq k).$$

Taking into account also that $\hat{C} = C$ and $\hat{C}Q = 0$, we find that $QCQ = (AQ)^*AQ = 0$ and consequently $AQ = 0$. Thus

$$A = \left(\sum_{j=1}^{\omega} P_j + Q\right) A \left(\sum_{j=1}^{\omega} P_j + Q\right) = \sum_{j=1}^{\omega} P_jAP_j = \hat{A},$$

and the theorem is proved.

THEOREM 5.2. *Suppose that the hypotheses of Theorem* 5.1 *are fulfilled, and let* $f(x)$ $(0 \leqq x < \infty; f(0) = 0)$ *be a nondecreasing convex function. Then*

$$\sum_{j=1}^{n} f(s_j(\hat{A})) \leqq \sum_{j=1}^{n} f(s_j(A)) \qquad (n = 1, 2, \cdots), \tag{5.7}$$

and consequently

$$\sum_{j=1}^{\infty} f(s_j(\hat{A})) \leqq \sum_{j=1}^{\infty} f(s_j(A)). \tag{5.8}$$

If the function $f(x)$ *is strictly convex, the equal sign in this relation* (*under the assumption that the right side is finite*) *will hold if and only if* $A = \hat{A}$.

PROOF. By Lemma 3.4, the inequalities (5.7) are consequences of the inequalities (5.2).

By the hypothesis of the final assertion of the theorem, the equal sign in (5.8) will hold, again on the basis of Lemma 3.4, only when the equalities (5.3) are fulfilled, and then, as we already know, $A = \hat{A}$. The theorem is proved.

REMARK 5.1. It is obvious that in the notation of Theorem 5.1 the inequality (5.8) can be written as

$$\sum_{k=1}^{\omega} \sum_{j=1}^{r_k} f(s_j(A_k)) \leqq \sum_{j=1}^{r(A)} f(s_j(A)).$$

REMARK 5.2. If concerning the function $f(x)$ $(0 \leqq x < \infty;\ f(0) = 0)$ one knows only that it is a *strictly increasing convex function* (but not necessarily strictly convex), one can still assert that the equal sign in (5.8) (under the assumption that the right side is finite) is possible only if the conditions

$$r(\hat{A}) \geqq r(A), \quad AQ = 0 \left(Q = I - \sum_{k=1}^{\omega} P_k \right)$$

are fulfilled.

The necessity of the first condition is obvious. To obtain the second condition, we put

$$B = A - AQ = A(I - Q). \tag{5.9}$$

Since

$$\hat{B} = \sum_{k=1}^{\omega} P_k B P_k = \sum_{k=1}^{\omega} P_k A P_k = \hat{A},$$

the inequality (5.8), applied to the operator B, yields the relation

$$\sum_{j=1}^{\omega} f(s_j(\hat{A})) \leqq \sum_{j=1}^{\omega} f(s_j(B)). \tag{5.10}$$

On the other hand, according to (2.3),

$$s_j(B) \leqq s_j(A) \qquad (j = 1, 2, \cdots), \tag{5.11}$$

and thus

$$\sum_{j=1}^{r(B)} f(s_j(B)) \leqq \sum_{j=1}^{r(A)} f(s_j(A)). \tag{5.12}$$

Therefore, if under the assumptions made with respect to $f(x)$ the equal sign holds in (5.8), then the equal sign holds in (5.10) and (5.12) and consequently also in all the relations (5.11). This means that $B^*B = A^*A$. Taking (5.9) into account, we obtain

$$QA^*AQ - A^*AQ - QA^*A = 0.$$

Multiplying this equality from left and right by Q, we obtain $AQ^*AQ = 0$; hence $AQ = 0$.

REMARK 5.3. Let $\{\phi_j\}_1^{\omega}$ ($\omega \leqq \infty$) be some orthonormal system of vectors, and P_j ($j = 1, 2, \cdots, \omega$) the operator of orthogonal projection in the direction of the unit vector ϕ_j ($j = 1, 2, \cdots, \omega$). It is easily seen that in this case the operator $A_j = P_j A P_j$ ($j = 1, 2, \cdots, \omega$) is either one-dimensional or equals zero, according as $(A\phi_j, \phi_j) \neq 0$ or $= 0$, and that

$$s_1(A_j) = |(A\phi_j, \phi_j)| \qquad (j = 1, 2, \cdots, \omega).$$

In this case the equality $A = \hat{A}$ indicates that (4.15) holds.

Thus all the assertions of §4.3 are special cases of Theorem 5.2.

§6. Inequalities for the eigenvalues of linear operators with completely continuous imaginary component

Let $A \in \mathfrak{R}$ be an operator with imaginary component $A_{\mathscr{J}} \in \mathfrak{S}_\infty$. According to Theorem I.5.2 the entire nonreal spectrum of A consists of not more than a countable number of spectral points, which are normal eigenvalues.

We shall denote by $\nu_{\mathscr{J}}(A)$ ($\leqq \infty$) the sum of the algebraic multiplicities of all the nonreal eigenvalues of the operator A. In this section we shall denote by $\lambda_j = \lambda_j(A)$ ($j = 1, 2, \cdots, \nu_{\mathscr{J}}(A)$) the system of all nonreal eigenvalues of the operator A ($A_{\mathscr{J}} \in \mathfrak{S}_\infty$), arbitrarily enumerated in order of decreasing modulus of their imaginary parts, taking into consideration algebraic multiplicity.

LEMMA 6.1. *Let A be a bounded linear operator with completely continuous imaginary component $A_{\mathscr{J}}$. Then*

$$\sum_{j=1}^{n} |\operatorname{Im}\lambda_j| \leqq \sum_{j=1}^{n} s_j(A_{\mathscr{J}}) \qquad (n = 1, 2, \cdots).$$

The equalities

$$|\operatorname{Im}\lambda_j| = s_j(A_{\mathscr{J}}) \qquad (j = 1, 2, \cdots, \nu_{\mathscr{J}}(A)) \tag{6.1}$$

will hold if and only if either 1) $\nu_{\mathscr{J}}(A) \geqq r(A_{\mathscr{J}})$ *and A is a normal operator, or* 2) $\nu_{\mathscr{J}}(A) < r(A_{\mathscr{J}})$ *(and hence $\nu_{\mathscr{J}}(A) < \infty$) and A has the following properties*:

a) *A induces a normal operator in the closed linear hull $\mathfrak{E}_{\mathscr{J}}$ of all the root subspaces corresponding to its nonreal eigenvalues*;

b) *the subspace $\mathfrak{H} \ominus \mathfrak{E}_{\mathscr{J}}$ is invariant with respect to A, and A induces in it an operator $\hat{A}$ with real spectrum, for which*

$$s_1(\hat{A}_{\mathscr{J}}) = s_{\nu_{\mathscr{J}}(A)+1}(A_{\mathscr{J}}).$$

PROOF. Consider an orthonormal Schur system ω_j $(j = 1, 2, \cdots, \nu_{\mathscr{J}}(A))$ of the operator A in the subspace $\mathfrak{E}_{\mathscr{J}}$. We have

$$A\omega_j = a_{j1}\omega_1 + a_{j2}\omega_2 + \cdots + a_{j,j-1}\omega_{j-1} + \lambda_j\omega_j \quad (j = 1, 2, \cdots, \nu_{\mathscr{J}}(A)). \tag{6.2}$$

Since

$$\lambda_j = (A\omega_j, \omega_j) = (A_{\mathscr{R}}\omega_j, \omega_j) + i(A_{\mathscr{J}}\omega_j, \omega_j),$$

it follows that

$$\operatorname{Im}\lambda_j = (A_{\mathscr{J}}\omega_j, \omega_j) \qquad (j = 1, 2, \cdots, \nu_{\mathscr{J}}(A)). \tag{6.3}$$

We denote by U a unitary operator such that

$$U\omega_j = \epsilon_j\omega_j \qquad (j = 1, 2, \cdots, \nu_{\mathscr{J}}(A)),$$

where $\epsilon_j = \operatorname{sign}(\operatorname{Im}\lambda_j)$ $(j = 1, 2, \cdots, \nu_{\mathscr{J}}(A))$. Then

$$|\operatorname{Im}\lambda_j| = (U^*A_{\mathscr{J}}\omega_j, \omega_j) \qquad (j = 1, 2, \cdots, \nu_{\mathscr{J}}(A)).$$

According to Lemma 4.1 we will have

$$\sum_{j=1}^{n} (U^*A_{\mathscr{J}}\omega_j, \omega_j) \leqq \sum_{j=1}^{n} s_j(A_{\mathscr{J}})$$

or

$$\sum_{j=1}^{n} |\operatorname{Im}\lambda_j| \leqq \sum_{j=1}^{n} s_j(A_{\mathscr{J}}).$$

Thus the first assertion of the lemma is proved.

Let us clarify when the relations (6.1) hold. It is obvious that if the operator A is normal or has the properties a), b), then the equalities (6.1) hold.

If (6.1) holds, then by (6.3)

$$(6.4) \qquad |(A_{\mathcal{J}}\omega_j, \omega_j)| = s_j(A_{\mathcal{J}}) \qquad (j = 1, 2, \cdots, \nu_{\mathcal{J}}(A)).$$

According to the arguments of the proof of Lemma 3.2, the latter signifies that the subspace $\mathfrak{E}_{\mathcal{J}}$ is invariant with respect to the operator $A_{\mathcal{J}}$, and the system of vectors $\{\omega_j\}$ is a complete orthonormal system of eigenvectors of the operator $A_{\mathcal{J}}$ in $\mathfrak{E}_{\mathcal{J}}$.

Bearing in mind that

$$(A\omega_j, \omega_k) = \overline{(A^*\omega_k, \omega_j)} = 0 \qquad (j < k)$$

we find that for the numbers a_{jk} in (6.2) we have

$$a_{jk} = (A\omega_j, \omega_k) = (A\omega_j, \omega_k) - (A^*\omega_j, \omega_k) = 2i(A_{\mathcal{J}}\omega_j, \omega_k) = 0 \quad (j > k).$$

Thus it follows from (6.4) that A induces a normal operator in $\mathfrak{E}_{\mathcal{J}}$.

The subspace $\mathfrak{E}_{\mathcal{J}}$ is obviously invariant with respect to the operator $A_{\mathcal{R}}$ $(= A - iA_{\mathcal{J}})$; consequently the subspace $\mathfrak{R} = \mathfrak{H} \ominus \mathfrak{E}_{\mathcal{J}}$ is invariant with respect to the operators $A_{\mathcal{J}}$, $A_{\mathcal{R}}$ and A.

It is easily verified that the operator $\hat{A}$, induced by A in $\mathfrak{R}$, has the properties a) and b).

If $\nu_{\mathcal{J}}(A) \geqq r(A_{\mathcal{J}})$, then by Lemma 3.2 it follows from (6.4) that the operator $A_{\mathcal{J}}$ annihilates the subspace $\mathfrak{R}$. Thus in this case the operator A generates a selfadjoint operator in $\mathfrak{R}$. The lemma is proved.

From this lemma and from Lemma 3.4 we obtain

THEOREM 6.1. *If $f(x)$ $(0 \leqq x < \infty;\ f(0) = 0)$ is a nondecreasing convex function, then for any bounded operator A with $A_{\mathcal{J}} \in \mathfrak{S}_\infty$*

$$\sum_{j=1}^{n} f(|\operatorname{Im} \lambda_j|) \leqq \sum_{j=1}^{n} f(s_j(A_{\mathcal{J}})) \qquad (n = 1, 2, \cdots, \nu_{\mathcal{J}}(A)).$$

In particular, if $\nu_{\mathcal{J}}(A) = \infty$, then

$$\sum_{j=1}^{\infty} f(|\operatorname{Im} \lambda_j|) \leqq \sum_{j=1}^{\infty} f(s_j(A_{\mathcal{J}})).$$

For a strictly convex function $f(x)$ the equality

$$\sum_{j=1}^{\nu_{\mathcal{J}}(A)} f(|\operatorname{Im} \lambda_j|) = \sum_{j=1}^{\infty} f(s_j(A_{\mathcal{J}})) \qquad (< \infty)$$

holds if and only if the operator A is normal.

Setting $f(x) = x^p$ $(1 \leq p < \infty)$ in this theorem, we obtain

$$\sum_{j=1}^{n} |\operatorname{Im} \lambda_j|^p \leqq \sum_{j=1}^{n} |\lambda_j(A_{\mathscr{I}})|^p \qquad (n = 1, 2, \cdots, \nu_{\mathscr{I}}(A)),$$

since $s_j(A_{\mathscr{I}}) = |\lambda_j(A_{\mathscr{I}})|$.

§7. s-numbers of bounded operators

1. To define the s-numbers of an arbitrary bounded linear operator A, we shall again put

$$s_j(A) = \lambda_j(H) \qquad (j = 1, 2, \cdots), \tag{7.1}$$

where $H = (A^*A)^{1/2}$. To give this definition meaning, however, it is necessary first to define the numbers $\lambda_j(H)$ $(j = 1, 2, \cdots)$ for any nonnegative operator $H \in \Re$.

A point λ of the spectrum of a selfadjoint operator $H \in \Re$ is called a point of the *condensed spectrum*, if it is either an accumulation point of the spectrum of H or an eigenvalue of H of infinite multiplicity.

Let A be any nonnegative operator from $\Re$ and let μ be the supremum of the spectrum of H. If the point μ belongs to the condensed spectrum of H, we put

$$\lambda_j(H) = \mu \qquad (j = 1, 2, \cdots).$$

If the point μ does not belong to the condensed spectrum of H, then it is an eigenvalue of finite multiplicity. In this case we put

$$\lambda_j(H) = \mu \qquad (j = 1, 2, \cdots, p),$$

where p is the multiplicity of the eigenvalue μ.

In the latter case the remaining numbers $\lambda_j(H)$ $(j = p + 1, \cdots)$ are defined by

$$\lambda_{p+j}(H) = \lambda_j(H_1) \qquad (j = 1, 2, \cdots),$$

where the operator H_1 is given by

$$H_1 = H - \mu P,$$

and P is the orthoprojector onto the eigenspace of the operator H corresponding to the eigenvalue μ.

The minimax properties[6] remain valid for the numbers $\lambda_j(H)$, so that

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[6] With min and max replaced, in the appropriate relations, by inf and sup.

$$\lambda_1(H) = \sup_{\phi \in \mathfrak{H}} \frac{(H\phi, \phi)}{(\phi, \phi)}$$

and

$$\lambda_{j+1} = \inf_{\mathfrak{L} \in \mathfrak{N}_j} \sup_{\phi \in \mathfrak{L}^{\perp}} \frac{(H\phi, \phi)}{(\phi, \phi)} \qquad (j = 1, 2, \cdots),$$

where $\mathfrak{N}_j$ $(j = 1, 2, \cdots)$ is the set of all j-dimensional subspaces of the space $\mathfrak{H}$.

From this one deduces at once that if $0 \leqq H_1 \leqq H_2$ $(H_1, H_2 \in \mathfrak{R})$, then

$$\lambda_j(H_1) \leqq \lambda_j(H_2) \qquad (j = 1, 2, \cdots).$$

The sequence $\{\lambda_j(H)\}_1^\infty$ is nonincreasing and thus has a limit. This limit $\lambda_\infty(H)$ is obviously the supremum of the condensed spectrum of the operator H.

If we make use of the spectral function $E(\lambda)$ $(= E(\lambda - 0))$ of the operator H, then a different, equivalent description can be given of the numbers $\lambda_j(H)$.

Let $\lambda_\infty(H)$ be the supremum of the condensed spectrum of H. The subspace $\mathfrak{H}_s = (I - E(\lambda_\infty(H) + 0))\mathfrak{H}$ is an invariant subspace of the operator H, and H induces in it an operator whose spectrum, with the possible exception of the point $\lambda_\infty(H)$, coincides with that part of the spectrum of H which lies in the interval $\lambda_\infty(H) \leqq \lambda < \infty$. Let us consider the operator $\hat{H}$, defined by

$$\hat{H}(\phi + \psi) = H\phi + \lambda_\infty(H)\psi \qquad (\phi \in \mathfrak{H}_s;\ \psi \in \mathfrak{H}_s^{\perp}).$$

The operator $\hat{H} - \lambda_\infty(H)I$ is completely continuous. It is easily seen that

$$\lambda_j(H) = \lambda_j(\hat{H} - \lambda_\infty(H)I) + \lambda_\infty(H) \qquad (j = 1, 2, \cdots).$$

We note further that $\hat{H}$ can be represented in the form

$$\hat{H} = \sum_j \lambda_j(H)(\cdot, \phi_j)\phi_j + \lambda_\infty(H)P_H, \tag{7.2}$$

where $\{\phi_j\}$ is an orthonormal basis of the subspace $\mathfrak{H}_s$, consisting of eigenvectors of H, and P_H is the orthoprojector which projects the space $\mathfrak{H}$ onto the subspace $\mathfrak{H}_s^{\perp}$; the series (7.2) converges strongly.

2. According to the definition (7.1), the sequence of s-numbers $\{s_j(A)\}_1^\infty$ of an arbitrary bounded operator A is nonincreasing and

$$\lim s_n(A) = s_\infty(A),$$

where $s_\infty(A)$ denotes the number $\lambda_\infty(H)$ $(H = (A^*A)^{1/2})$.

From the definition of the s-numbers it follows at once that

$$s_j(A) = s_j(A^*) \qquad (A \in \mathfrak{R};\ j = 1, 2, \cdots)$$

and for any scalar c

$$s_j(cA) = |c|\, s_j(A) \qquad (A \in \mathfrak{R};\ j = 1, 2, \cdots).$$

From the minimax properties of the numbers $\lambda_j(H)$ one easily deduces the relations

$$s_j(BA) \leqq |B|\, s_j(A);\ s_j(AB) \leqq |B|\, s_j(A) \qquad (A, B \in \mathfrak{R};\ j = 1, 2, \cdots).$$

Let $A = UH$ be the polar representation of some operator $A \in \mathfrak{R}$; let us replace the operator H by the operator $\hat{H}$, defined by (7.2), and put $\hat{A} = U\hat{H}$. Obviously the operator $\hat{A}$ has the same s-numbers as A. We represent the operator $\hat{A}$ in the form

$$\hat{A} = \sum_j s_j(A)\,(\cdot, \phi_j)\,\psi_j + s_\infty(A)\, UP_H, \tag{7.3}$$

where $\{\phi_j\}$ is the orthonormal system from (7.2), and $\psi_j = U\phi_j$. We shall call the series (7.3) the *Schmidt series* of the operator $\hat{A}$, and the operator

$$K_n = \sum_{j=1}^{n} s_j(A)\,(\cdot, \phi_j)\,\psi_j$$

the *nth partial Schmidt series* of the operator $\hat{A}$.

We note further that the operator $\hat{A} - s_\infty(A)\,U$ is completely continuous.

3. Theorem 2.1 on an approximation property of the s-numbers of completely continuous operators and its corollaries can be extended to the s-numbers of bounded operators.

THEOREM 7.1. *Let A be any operator from $\mathfrak{R}$; then for any $n = 0, 1, 2, \cdots$*

$$s_{n+1}(A) = \min_{K \in \mathfrak{K}_n} |A - K|. \tag{7.4}$$

This theorem shows the naturalness of the introduction of s-numbers for an arbitrary bounded operator. Just as for completely continuous operators, the equality (7.4) can be taken as a new equivalent definition of the s-numbers.

PROOF. Just as in the proof of Theorem 2.1 we can show that

$$s_{n+1}(A) \leqq |A - K| \qquad (K \in \mathfrak{K}_n). \tag{7.5}$$

If the number $s_n(A)$, and consequently all the preceding ones, are

eigenvalues of the operator $H = (A^*A)^{1/2}$, then it is easily seen that

$$|A - K_n| = s_{n+1}(A),$$

where K_n is the nth partial Schmidt series of the operator $\hat{A}$. Thus for this case the theorem is proved.

Let us consider the case in which the spectral point $\lambda = s_n(A)$ of the operator H is not an eigenvalue of H. We denote by p $(0 \leqq p \leqq n)$ the smallest number for which $s_p(A)$ is an eigenvalue of the operator H. Then obviously

$$s_j(A) = s_\infty(A) \qquad (j = p+1, p+2, \cdots).$$

If $p \neq 0$ then, as was proved,

$$s_{p+1}(A) = |A - K_p|,$$

and consequently

$$s_{n+1}(A) = |A - K_p| \qquad (K_p \in \mathfrak{K}_p \subset \mathfrak{K}_n). \tag{7.6}$$

Comparing (7.5) and (7.6), we obtain (7.4).

Finally, if $p = 0$, then

$$s_j(A) = s_\infty(A) = |A| \qquad (j = 1, 2, \cdots). \tag{7.7}$$

By virtue of (7.5) the last equality holds if and only if

$$\min_{K \in \mathfrak{K}_n} |A - K| = |A| \qquad (n = 1, 2, \cdots).$$

The theorem is proved.

Corollaries 2.1 and 2.2 of Theorem 2.1 extend word for word to an arbitrary bounded operator. Let us note yet another corollary of Theorem 7.1.

COROLLARY 7.1. *If* $A \in \mathfrak{R}$, *then*

$$\min_{T \in \mathfrak{S}_\infty} |A - T| = s_\infty(A).$$

Indeed, using Theorem 7.1 and the fact that any operator from $\mathfrak{S}_\infty$ can be approximated arbitrarily closely in the uniform norm by finite-dimensional operators, we have

$$\inf_{T \in \mathfrak{S}_\infty} |A - T| = \lim_{n \to \infty} \min_{K \in \mathfrak{K}_n} |A - K| = \lim_{n \to \infty} s_n(A) = s_\infty(A).$$

Moreover, it is obvious that for $T = \hat{A} - s_\infty(A)U$

$$|A - T| = s_\infty(A).$$

Thus the number $s_\infty(A)$ is the distance from the operator A to $\mathfrak{S}_\infty$, the subspace of all completely continuous operators.

Without going into details, we mention that Lemma 4.1 carries over word for word to bounded operators, with the sole difference that in the relation (4.1) max is replaced by sup. From this lemma one easily deduces that for any bounded operators $A, B \in \mathfrak{R}$

$$\sum_{j=1}^{n} s_j(A+B) \leqq \sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B) \qquad (n = 1, 2, \cdots). \tag{7.8}$$

For any bounded operators A, B the relation

$$\sum_{j=1}^{n} s_j(AB) \leqq \sum_{j=1}^{n} s_j(A) s_j(B) \qquad (n = 1, 2, \cdots) \tag{7.9}$$

also remains valid.

We prove the last assertion. Let the polar representation of the operator AB have the form $AB = UH$. We denote by p ($\leqq \infty$) the smallest number for which $s_{p+1}(AB) = s_\infty(AB)$, and by ϕ_j ($j = 1, 2, \cdots, p$) an orthonormal system of eigenvectors of the operator H, corresponding to its eigenvalues $s_j(AB)$ ($j = 1, 2, \cdots, p$). With every positive number ϵ and positive integer n we associate the subspace $\mathfrak{N}_{\epsilon;n} = E(s_\infty(AB) + 0)\mathfrak{H} \ominus E(s_\infty(AB) - \epsilon/n)\mathfrak{H}$, where $E(\lambda)$ is the spectral function of the operator H.

With any $n \leqq p$ we associate the operator

$$H_n = \sum_{j=1}^{n} s_j(AB)(\cdot, \phi_j)\phi_j. \tag{7.10}$$

If $p < \infty$, the subspace $\mathfrak{N}_{\epsilon;n}$ is infinite dimensional. In this case we define the operator H_n for $n > p$ by

$$H_n = \sum_{j=1}^{p} s_j(AB)(\cdot, \phi_j)\phi_j + \sum_{j=p+1}^{\infty} s_\infty(AB)(\cdot, \phi_j)\phi_j, \tag{7.11}$$

where ϕ_j ($j = p+1, p+2, \cdots$) is an arbitrary orthonormal system of vectors of the subspace $\mathfrak{N}_{\epsilon;n}$. Finally, we denote by P_n the orthoprojector which projects the entire space $\mathfrak{H}$ onto the subspace $\mathfrak{M}_{\epsilon;n}$ with basis $\{\phi_j\}_1^n$.

It follows at once from the definition of the operator H_n that for $n > p$ and for all vectors $f \in \mathfrak{M}_{\epsilon;n}$

$$|Hf - H_n f| \leqq \epsilon |f|/n.$$

It follows from (7.10) and (7.11) that for any positive integer n one has the relation

$$|P_nHP_n - H_n| \leqq |(H - H_n)P_n| \leqq \epsilon/n. \tag{7.12}$$

Since $s_j(AB) = s_j(H_n)$ $(j = 1, 2, \cdots, n)$, according to the relations (7.8) and (7.12)

$$\begin{aligned} \sum_{j=1}^{n} s_j(AB) = \sum_{j=1}^{n} s_j(H_n) &\leqq \sum_{j=1}^{n} s_j(P_nHP_n) + \sum_{j=1}^{n} s_j(P_nHP_n - H_n) \\ &\leqq \sum_{j=1}^{n} s_j(P_nHP_n) + \epsilon. \end{aligned} \tag{7.13}$$

Now $P_nHP_n = A_1B_1$, where $A_1 = P_nU^*A$ and $B_1 = BP_n$. The operators A_1 and B_1 are finite-dimensional, and thus, according to (4.9),

$$\sum_{j=1}^{n} s_j(A_1B_1) \leqq \sum_{j=1}^{n} s_j(A_1)\, s_j(B_1). \tag{7.14}$$

Bearing in mind that $s_j(A_1) \leqq s_j(A)$ and $s_j(B_1) \leqq s_j(B)$, we obtain from (7.13) and (7.14) the relation

$$\sum_{j=1}^{n} s_j(AB) \leqq \sum_{j=1}^{n} s_j(A)\, s_j(B) + \epsilon.$$

Since ϵ is arbitrary, (7.9) follows.

In this section we have considered only those properties of the s-numbers of bounded operators which we will need later. However, many other results concerning the s-numbers of completely continuous operators can be extended to the s-numbers of bounded operators.

CHAPTER III

SYMMETRICALLY-NORMED IDEALS OF THE RING OF BOUNDED LINEAR OPERATORS

A fundamental role will be played in the sequel by various classes of completely continuous operators. As examples of such classes we can name the class $\mathfrak{S}_1$ of nuclear operators, the class $\mathfrak{S}_2$ of Hilbert-Schmidt operators and their natural generalization—the classes $\mathfrak{S}_p$ $(p > 0)$. These classes are "veterans" at the present time, which have been adequately studied and are encountered in many investigations.

However, recent investigations in the problem of reducing a nonselfadjoint operator to triangular form have made clear the necessity of introducing certain new classes of completely continuous operators. The analysis of these classes has revealed that all of them can be included in the framework of the theory of "cross-spaces," developed by J. von Neumann and R. Schatten (see R. Schatten [1]).

This theory, applied to operators acting in a Hilbert space, is discussed in a separate monograph by R. Schatten [2] under the name of the theory of "norm-ideals."

The present chapter includes the basic content of this monograph. In this chapter we also discuss a number of essential complements to the theory of von Neumann-Schatten. Complements to the general theory are contained in §§5 and 6. New classes of concrete norm-ideals are studied in §§14 and 15. Some of these are the first examples of nonseparable norm-ideals, consisting of completely continuous operators in a separable Hilbert space. The interpolation theorems on wandering operator-functions (§§13 and 16) are new. Also new are the methods for obtaining the general results of the theory. In distinction to the methods of von Neumann and Schatten, they are based on the repeated use of the properties of the s-numbers of completely continuous operators.

In this book, "norm-ideals" are named *symmetrically-normed ideals,* abbreviated *s.n. ideals*. At the same time it must be emphasized that in writing this chapter, essential use has been made of R. Schatten's monograph [2]. We particularly mention that, in view of the importance of the class of nuclear operators, we give in a special section (§10) analytic

criteria for the nuclearity of integral operators and a rule for calculating the trace of such operators.

§1. Two-sided ideals of the ring of bounded linear operators

1. According to the general definition from the theory of rings, a set $\mathfrak{M} \subset \mathfrak{R}$ is called an (algebraic) *two-sided ideal* of the ring $\mathfrak{R}$, if it has the following properties:

a) for any operators $A, B \in \mathfrak{M}$ we have $A + B \in \mathfrak{M}$;

b) for any $A \in \mathfrak{M}$ and any $B \in \mathfrak{R}$ we have $AB, BA \in \mathfrak{M}$;

c) $\mathfrak{M} \neq \{0\}$ and $\mathfrak{M} \neq \mathfrak{R}$.

Since $I \in \mathfrak{R}$ and $\mathfrak{R}$ is a linear set, every ideal $\mathfrak{M}$ is a linear set.

Obviously the set $\mathfrak{K}$ of all finite-dimensional operators from $\mathfrak{R}$ is a two-sided ideal of the ring $\mathfrak{R}$.

The set $\mathfrak{S}_\infty$ of all completely continuous operators is also a two-sided ideal of the ring $\mathfrak{R}$. It turns out that $\mathfrak{K}$ is the minimal and $\mathfrak{S}_\infty$ the maximal two-sided ideal of the ring $\mathfrak{R}$. We have the following theorem.

THEOREM 1.1 (J. CALKIN [1]). *Any two-sided ideal $\mathfrak{M}$ of the ring $\mathfrak{R}$ is contained in $\mathfrak{S}_\infty$ and contains $\mathfrak{K}$:*

$$\mathfrak{K} \subseteq \mathfrak{M} \subseteq \mathfrak{S}_\infty.$$

PROOF. We assume that the ideal $\mathfrak{M}$ contains at least one operator A which is not completely continuous, and we shall prove that $\mathfrak{M}$ will then contain an invertible operator. Let the polar representation of the operator A have the form $A = UH$. Then the operator $H = U^*A$ belongs to the ideal $\mathfrak{M}$ and does not belong to $\mathfrak{S}_\infty$. Let us denote by $E(\lambda)$ the spectral function of the operator H.

Since $H \notin \mathfrak{S}_\infty$, one can find $\epsilon > 0$ for which the orthoprojector

$$P = E(\infty) - E(\epsilon)$$

is infinite-dimensional. The subspace $P\mathfrak{H}$ is invariant with respect to H, and H induces in it an invertible operator $\hat{H}$.

We denote by V some operator which maps the space $\mathfrak{H}$ isometrically onto the subspace $P\mathfrak{H}$. The operator V^* maps the subspace $P\mathfrak{H}$ isometrically onto all of $\mathfrak{H}$, and

$$V^*V = I, \qquad VV^* = P.$$

Then $V^*HV = V^*\hat{H}V \in \mathfrak{M}$ and is invertible: $(V^*HV)^{-1} = V^*\hat{H}^{-1}V$.

Consequently, the unit operator I appears in the ideal $\mathfrak{M}$, and with it the entire ring $\mathfrak{R}$. This is not possible. Therefore $\mathfrak{M} \subseteq \mathfrak{S}_\infty$.

To prove the inclusion $\mathfrak{K} \subseteq \mathfrak{M}$, it is obviously sufficient to show that every one-dimensional operator is contained in $\mathfrak{M}$.

Let $K = (\cdot, \phi)\psi$ be any one-dimensional operator, and let A be any nonzero operator from $\mathfrak{M}$. It is easily seen that one can find operators $B, C \in \mathfrak{R}$ such that

$$BAC\psi = \psi.$$

Now $BAC \in \mathfrak{M}$, and, moreover,

$$BACK = K.$$

Thus $K \in \mathfrak{M}$. The theorem is proved.

COROLLARY 1.1. *The ideal* $\mathfrak{S}_\infty$ *is the only closed two-sided ideal of the ring* $\mathfrak{R}$.

2. We cite two further properties of ideals of the ring $\mathfrak{R}$.

1. *If the operator A belongs to a two-sided ideal $\mathfrak{M}$ of the ring $\mathfrak{R}$, then the operator A^* also belongs to this ideal; in other words, every two-sided ideal is selfadjoint.*

Let the polar representation of the operator A have the form $A = UH\,(H = (A^*A)^{1/2})$. Then, as was already noted, $H = U^*A \in \mathfrak{M}$ and consequently the operator $A^* = HU^*$ belongs to the ideal $\mathfrak{M}$.

If $A \in \mathfrak{R}$ and λ^{-1} is a regular point of A, then the operator $A(\lambda)$ $(\in \mathfrak{R})$, defined by

$$I + \lambda A(\lambda) = (I - \lambda A)^{-1},$$

is called the *Fredholm resolvent* of the operator A.

It is obvious that for $|\lambda| < |A|^{-1}$ the Fredholm resolvent $A(\lambda)$ admits the expansion

$$A(\lambda) = \sum_{j=0}^{\infty} \lambda^j A^{j+1},$$

which is convergent in the uniform norm. From the equality

$$(I + \lambda A(\lambda))(I - \lambda A) = (I - \lambda A)(I + \lambda A(\lambda)) = I$$

follows

$$A(\lambda) = A + \lambda A A(\lambda) = A + \lambda A(\lambda) A,$$

which at once implies the result

2. *If the operator A belongs to some two-sided ideal $\mathfrak{M}$ of the ring $\mathfrak{R}$, then its Fredholm resolvent also belongs to $\mathfrak{M}$.*

§ 2. Symmetrically-normed ideals

1. To define symmetrically-normed ideals of the ring $\Re$, we need the concept of a symmetric norm. A functional $|X|_s$, defined on some two-sided ideal $\mathfrak{S}$ of the ring $\Re$, is called a *symmetric norm* if it has the usual properties of a norm:

1) $|X|_s > 0$ $(X \in \mathfrak{S};\ X \neq 0)$;

2) $|\lambda X|_s = |\lambda|\ |X|_s$ $(X \in \mathfrak{S})$, where λ is any complex number;

3) $|X + Y|_s \leqq |X|_s + |Y|_s$ $(X, Y \in \mathfrak{S})$;

and if, moreover,

4) $|AXB|_s \leqq |A|\ |X|_s |B|$ $(A, B \in \Re;\ X \in \mathfrak{S})$;

5) for any one-dimensional operator X[1]

$$|X|_s = |X| \equiv s_1(X).$$

Obviously the ordinary operator norm on any ideal $\mathfrak{S}$ is a symmetric norm.

If, in the definition of a symmetric norm, the condition 4) is replaced by the condition

4′) $|UX|_s = |XU|_s = |X|_s$ $(X \in \mathfrak{S})$,

where U is any unitary operator, then we obtain the definition of an *invariant norm*.

Every symmetric norm is invariant.[2] In fact, according to the property 4) we will have, for any unitary operators U, V,

$$|UXV|_s \leqq |X|_s \qquad (X \in \mathfrak{S}).$$

On the other hand, since $X = U^{-1}UXVV^{-1}$, we have

$$|X|_s \leqq |UXV|_s.$$

Thus

$$|UXV|_s = |X|_s.$$

A symmetric norm has a number of important properties.

1. *Let $\mathfrak{S}$ be some two-sided ideal of the ring $\Re$ and let a symmetric norm be defined on $\mathfrak{S}$. Then for any operator $X \in \mathfrak{S}$*

$$|X|_s = |X^*|_s = |(XX^*)^{1/2}|_s = |(X^*X)^{1/2}|_s.$$

In fact, let $X = UH$ be the polar representation of the operator X. Then

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[1] It is easily seen that condition 4) implies $|X|_s = a|X|$ for any one-dimensional X; here a is a positive constant not depending upon X. Thus condition 5) is a "normalization" condition for a norm having the property 4).

[2] In the following sections it will be shown that this result admits a converse if the ideal $\mathfrak{S}$ is separable.

$$|X|_s \leqq |H|_s.$$

Since $U^*X = H$,

$$|H|_s \leqq |X|_s.$$

Consequently $|X|_s = |H|_s$.

Starting from the equalities $X^* = HU^*$ and $X^*U = H$, we obtain in the same way $|X^*|_s = |H|_s$.

For an ideal $\mathfrak{S}$ satisfying the hypotheses of the result 1, result 2 is also valid:

2. *Let $X \in \mathfrak{S}$; then every operator Y $(\in \mathfrak{S}_\infty)$ for which*

$$s_j(Y) \leqq cs_j(X) \qquad (j = 1, 2, \cdots), \tag{2.1}$$

where c is a positive constant, belongs to $\mathfrak{S}$, and

$$|Y|_s \leqq c|X|_s. \tag{2.2}$$

In fact, if $H_x = (X^*X)^{1/2}$ and $H_y = (Y^*Y)^{1/2}$, then by virtue of (2.1) one can find a unitary operator V and a nonnegative operator $A \in \mathfrak{R}$ with $|A| \leqq 1$ such that

$$H_y = cAVH_xV^{-1}. \tag{2.3}$$

For the operator V we choose an operator which maps some orthonormal basis of eigenvectors of the operator H_x into an appropriate orthonormal basis of eigenvectors of H_y. It follows fiom (2.3) that $H_y \in \mathfrak{S}$ and $|H_y|_s \leqq c|H_x|_s$. Hence it follows at once that $Y \in \mathfrak{S}$ and that the relation (2.2) is fulfilled.

3. *For any symmetric norm $|X|_s$ defined on some ideal $\mathfrak{S}$ we have*

$$s_1(X) \leqq |X|_s,$$

and if $\dim X < \infty$, *then also*

$$|X|_s \leqq \sum_j s_j(X).$$

In fact, let $Y = s_1(X)(\cdot, \phi)\phi$, where ϕ is an arbitrary unit vector of $\mathfrak{H}$. Then the relations (2.1) are fulfilled for the operators Y and X with $c = 1$, and so

$$s_1(X) = |X| = |Y| = |Y|_s \leqq |X|_s. \tag{2.4}$$

Let

$$X = \sum_j s_j(X)(\cdot, \psi_j)\phi_j$$

be the Schmidt expansion of the operator X. From it, on the basis of

properties 3) and 5), we obtain

$$|X|_s \leqq \sum_j s_j(X).$$

2. We shall call a two-sided ideal $\mathfrak{S}$ of the ring $\mathfrak{R}$ a *symmetrically-normed ideal* of the ring $\mathfrak{R}$, if there is defined on it a symmetric norm $|X|_{\mathfrak{S}}$ which makes $\mathfrak{S}$ a Banach space. For brevity we shall call a symmetrically-normed ideal an *s.n. ideal.*[3)]

We shall say that two ideals $\mathfrak{S}_{\mathrm{I}}$ and $\mathfrak{S}_{\mathrm{II}}$ *coincide elementwise*, if $\mathfrak{S}_{\mathrm{I}}$ and $\mathfrak{S}_{\mathrm{II}}$ consist of the same elements.

THEOREM 2.1. *If the s.n. ideals $\mathfrak{S}_{\mathrm{I}}$ and $\mathfrak{S}_{\mathrm{II}}$ coincide elementwise, then their norms are topologically equivalent.*

PROOF. Let us denote by $\mathfrak{S}$ the set of elements of the s.n. ideal $\mathfrak{S}_{\mathrm{I}}$ or, what is the same, of the s.n. ideal $\mathfrak{S}_{\mathrm{II}}$. We consider the functional $\|X\|$ $(X \in \mathfrak{S})$ defined by

$$\|X\| = \max\{|X|_{\mathrm{I}}, |X|_{\mathrm{II}}\},$$

where $|X|_{\mathrm{I}} = |X|_{\mathfrak{S}_{\mathrm{I}}}$ and $|X|_{\mathrm{II}} = |X|_{\mathfrak{S}_{\mathrm{II}}}$. This functional has all the properties of a norm.

From the inequalities

$$|X| \leqq |X|_{\mathrm{I}} \quad \text{and} \quad |X| \leqq |X|_{\mathrm{II}} \qquad (X \in \mathfrak{S})$$

it follows that if a sequence of operators converges in each of the norms $|X|_{\mathrm{I}}$ and $|X|_{\mathrm{II}}$, then in each of these norms it has the same limit. It follows at once that $\mathfrak{S}$ is complete in the new norm $\|X\|$. Let us consider the identity mapping I as an operator, acting from the space $\mathfrak{S}$ with norm $\|X\|$ into each of the spaces $\mathfrak{S}_{\mathrm{I}}$ and $\mathfrak{S}_{\mathrm{II}}$. It follows from the definition of the norm $\|X\|$ that I is a bounded operator with norm $\leqq 1$. But then, by a well-known theorem of S. Banach [1], the operator I is a bicontinuous mapping from each of the spaces $\mathfrak{S}_{\mathrm{I}}$ and $\mathfrak{S}_{\mathrm{II}}$ onto the space $\mathfrak{S}$ with norm $\|X\|$. Hence each of the norms $|X|_{\mathrm{I}}$, $|X|_{\mathrm{II}}$ is topologically equivalent to the norm $\|X\|$, and so the norms $|X|_{\mathrm{I}}$ and $|X|_{\mathrm{II}}$ are topologically equivalent to each other.

3. It follows at once from result 2 that every symmetric norm $|X|_s$ depends only on the s-numbers of the operator X. (In other words, if the s-numbers of the operators X_1 and X_2 coincide, then their norms $|X_1|_s$ and $|X_2|_s$ coincide.)

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3) A new approach to the study of s.n. ideals, based on the theory of categories, is indicated in a paper of B. S. Mitjagin and A. S. Švarc [1].

Thus for every symmetric norm we have

$$|X|_s = \Phi(s_1(X), s_2(X), \cdots), \tag{2.5}$$

where $\Phi(\xi_1, \xi_2, \cdots)$ is a function of the nonnegative variables ξ_j which has determinable properties.

An important case is that in which $\mathfrak{S}$ coincides with the ideal $\mathfrak{K}$ of finite-dimensional operators; in this case the domain of the function (2.5) consists of all nonincreasing sequences $\{\xi_j\}$ of nonnegative numbers, of which only finitely many are different from zero.

The following section is devoted to the study of functions $\Phi(\xi_1, \xi_2, \cdots)$ which define a symmetric norm on the ideal $\mathfrak{K}$ (these functions will be called symmetric norming functions).

§ 3. Symmetric norming functions

1. Let c_0 be the space of all sequences $\xi = \{\xi_j\}_1^\infty$ of real numbers which tend to zero. We denote by $\hat{c}$ the lineal of c_0 consisting of all sequences with a finite number of nonzero terms.

A real function $\Phi(\xi) = \Phi(\xi_1, \xi_2, \cdots)$, defined on the lineal $\hat{c}$, is called a *norming function* if it has the following properties:

I) $$\Phi(\xi) > 0 \qquad (\xi \in \hat{c},\ \xi \neq 0);$$

II) for any real α

$$\Phi(\alpha\xi) = |\alpha|\Phi(\xi) \qquad (\xi \in \hat{c});$$

III) $$\Phi(\xi + \eta) \leqq \Phi(\xi) + \Phi(\eta) \qquad (\xi, \eta \in \hat{c});$$

IV) $$\Phi(1, 0, 0, \cdots) = 1.$$

A norming function $\Phi(\xi)$ is said to be *symmetric* if it has the property

V) $$\Phi(\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots) = \Phi(|\xi_{j_1}|, |\xi_{j_2}|, \cdots, |\xi_{j_n}|, 0, 0, \cdots),$$

where $\xi = \{\xi_j\}$ is any vector from $\hat{c}$, and $j_1, j_2, \cdots, j_n$ is any permutation of the integers $1, 2, \cdots, n$.

We cite various properties of symmetric norming functions (s. n. functions).

1. *If the relations*

$$|\xi_j| \leqq |\eta_j| \qquad (j = 1, 2, \cdots)$$

hold for the vectors $\xi = \{\xi_j\}$, $\eta = \{\eta_j\} \in \hat{c}$, *then*

$$\Phi(\xi) \leqq \Phi(\eta). \tag{3.1}$$

In fact, we may assume without loss of generality that the coordinates

ξ_j and η_j are nonnegative. Obviously the result will be proved as soon as it is proved for the case

$$\xi_j = \eta_j \quad (j \neq k), \quad \xi_k < \eta_k,$$

where k is some positive integer. Let us denote by α the ratio ξ_k/η_k. Then

$$\Phi(\xi) = \Phi(\xi' + \xi'') \leqq \Phi(\xi') + \Phi(\xi''), \tag{3.2}$$

where

$$\xi_j' = \frac{1+\alpha}{2}\xi_j, \quad \xi_j'' = \frac{1-\alpha}{2}\xi_j \qquad (j \neq k)$$

and

$$\xi_k' = \frac{1+\alpha}{2}\eta_k, \qquad \xi_k'' = -\frac{1-\alpha}{2}\eta_k.$$

Since

$$\Phi(\xi') = \frac{1+\alpha}{2}\Phi(\eta),$$

and by virtue of the condition V)

$$\Phi(\xi'') = \frac{1-\alpha}{2}\Phi(\xi_1, \xi_2, \cdots, -\eta_k, \cdots) = \frac{1-\alpha}{2}\Phi(\eta),$$

the relation (3.1) follows from (3.2).

LEMMA 3.1 (K. FAN [3]). *Suppose* $\xi = \{\xi_j\}$ *and* $\eta = \{\eta_j\} \in \hat{c}$. *If*

$$\xi_1 \geqq \xi_2 \geqq \cdots \geqq 0; \qquad \eta_1 \geqq \eta_2 \geqq \cdots \geqq 0$$

and

$$\sum_{j=1}^{k}\xi_j \leqq \sum_{j=1}^{k}\eta_j \qquad (k = 1, 2, \cdots),$$

then for any symmetric norming function $\Phi(\xi)$ *one has*

$$\Phi(\xi) \leqq \Phi(\eta).$$

The proof of this lemma is based upon the following result concerning vectors of n-dimensional real space.

Suppose that the vectors $\xi = \{\xi_j\}_1^n$ *and* $\eta = \{\eta_j\}_1^n$ *satisfy the conditions*

$$\xi_1 \geqq \xi_2 \geqq \cdots \geqq \xi_n \geqq 0; \quad \eta_1 \geqq \eta_2 \geqq \cdots \geqq \eta_n \geqq 0 \tag{3.3}$$

and

$$\sum_{j=1}^{k}\xi_j \leqq \sum_{j=1}^{k}\eta_j \qquad (k = 1, 2, \cdots, n). \tag{3.4}$$

Then the vector ξ *admits the representation*

(3.5) $$\xi = \sum_{\nu=1}^{N} t_\nu \eta^{(\nu)} \qquad (N = 2^n n!),$$

where the $\eta^{(\nu)}$ $(\nu = 1, 2, \cdots, N)$ *are all the n-dimensional vectors obtainable from* η *by permuting its coordinates and multiplying them by* ± 1, *and the* t_ν *are nonnegative numbers with sum* $\sum_{\nu=1}^{N} t_\nu = 1$.

In other words, every vector ξ satisfying conditions (3.3) and (3.4) belongs to the convex hull of the system of vectors $\eta^{(\nu)}$.

This result is due to A. S. Markus [4]. The proof presented here is by B. S. Mitjagin [1].

Let us denote by G the convex hull of all the vectors $\eta^{(\nu)}$, and assume that $\xi \notin G$. Since every convex body is the intersection of its supporting half-spaces, one can find a supporting hyperplane of G such that the vector ξ and G lie on opposite sides of it. Obviously, the equation $\sum_{j=1}^{n} a_j x_j = b$ of this hyperplane can be written so that for any vector $\{x_j\}_1^n$ from G the relation

$$\sum_{j=1}^{n} a_j x_j \leqq b$$

will hold, and for the vector $\xi = \{\xi_j\}$ we will have

$$\sum_{j=1}^{n} a_j \xi_j > b.$$

From among the vectors $\eta^{(\nu)}$ one can choose a vector $\eta^{(r)} = \{\eta_j^{(r)}\}$ such that

$$\sum_{j=1}^{n} a_j \eta_j^{(r)} = \sum_{j=1}^{n} a_j^* \eta_j,$$

where $\{a_j^*\}_1^n$ is the vector with components equal to $|a_1|, |a_2|, \cdots, |a_n|$ and arranged in nonincreasing order. Since $\eta^{(r)} \in G$, we have

$$\sum_{j=1}^{n} a_j^* \eta_j \leqq b.$$

Taking into account the equality

$$\sum_{j=1}^{n} a_j^* \eta_j = \sum_{j=1}^{n} \eta_j a_n^* + \sum_{j=1}^{n-1} (a_j^* - a_{j+1}^*) \sum_{k=1}^{j} \eta_k,$$

we obtain, by virtue of (3.4),

$$\sum_{j=1}^{n} a_j^* \eta_j \geqq \sum_{j=1}^{n} a_j^* \xi_j.$$

Consequently

$$\sum_{j=1}^{n} a_j^* \xi_j \leqq b. \tag{3.6}$$

On the other hand, from the equality

$$\sum_{j=1}^{n} a_j^* \xi_j = \sum_{j=1}^{n} a_j^* \xi_n + \sum_{j=1}^{n-1} (\xi_j - \xi_{j+1}) \sum_{k=1}^{j} a_k^*$$

and the obvious relations

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} a_j^* \qquad (k = 1, 2, \cdots, n)$$

it follows that

$$\sum_{j=1}^{n} a_j^* \xi_j \geqq \sum_{j=1}^{n} a_j \xi_j > b. \tag{3.7}$$

The relations (3.6) and (3.7) are mutually contradictory, which proves the result.

PROOF OF LEMMA 3.1. Suppose that all the coordinates of the vectors ξ and η with index greater than n equal zero. Let us denote by $\eta^{(\nu)}$ vectors from $\hat{c}$ such that their first n coordinates are obtained from the numbers η_j $(j = 1, 2, \cdots, n)$ by permutations and by multiplication by ± 1. Then according to (3.5)

$$\xi = \sum_{\nu} t_\nu \eta^{(\nu)} \quad \text{and} \quad \sum_{\nu} t_\nu = 1, \qquad t_\nu \geqq 0.$$

Since

$$\Phi(\xi) \leqq \sum_{\nu} t_\nu \Phi(\eta^{(\nu)}),$$

and $\Phi(\eta^{(\nu)}) = \Phi(\eta)$, it follows that $\Phi(\xi) \leqq \Phi(\eta)$. The lemma is proved.

2. Let us denote by $\hat{k}$ the cone of all nonincreasing sequences from $\hat{c}$, each of which consists of nonnegative numbers. With every vector $\xi = \{\xi_j\} \in \hat{c}$ we associate the vector $\xi^* = \{\xi_j^*\} \in \hat{k}$, setting

$$\xi_j^* = |\xi_{n_j}| \qquad (j = 1, 2, \cdots),$$

where n_j $(j = 1, 2, \cdots)$ is a permutation of the positive integers such that the sequence $\{|\xi_{n_j}|\}$ is nonincreasing.

Since for any s.n. function $\Phi(\xi)$ one has

$$\Phi(\xi) = \Phi(\xi^*) \qquad (\xi \in \hat{c}), \tag{3.8}$$

an s.n. function $\Phi(\xi)$ is uniquely defined by its values on the cone $\hat{k}$. It

follows that the conditions I)—V) defining an s. n. function can be replaced by equivalent conditions in which only vectors from $\hat{k}$ appear.

LEMMA 3.2. *Let $\Phi(\xi)$ be a function, defined on the cone $\hat{k}$. In order that the equality*

$$\Phi(\xi) = \Phi(\xi^*) \qquad (\xi \in \hat{c}) \tag{3.9}$$

define an s.n. function, it is necessary and sufficient that the following conditions be satisfied:

I') $$\Phi(\xi) > 0 \qquad (\xi \in \hat{k};\ \xi \neq 0);$$

II') *for any nonnegative number* α

$$\Phi(\alpha\xi) = \alpha\Phi(\xi) \qquad (\xi \in \hat{k});$$

III') $$\Phi(\xi + \eta) \leqq \Phi(\xi) + \Phi(\eta) \qquad (\xi, \eta \in \hat{k});$$

IV') $$\Phi(1, 0, 0, \cdots) = 1;$$

V') *if* $\xi = \{\xi_j\}$, $\eta = \{\eta_j\} \in \hat{k}$ *and*

$$\sum_{j=1}^{n} \xi_j \leqq \sum_{j=1}^{n} \eta_j \qquad (n = 1, 2, \cdots),$$

then

$$\Phi(\xi) \leqq \Phi(\eta).$$

PROOF. Every s. n. function $\Phi(\xi)$, by definition, has the properties I')—IV'). Moreover, according to Lemma 3.1 it also has property V').

Conversely, let the function $\Phi(\xi)$, defined on the cone $\hat{k}$, have the properties I')—V'); then the function $\Phi(\xi)$, defined on the lineal $\hat{c}$ by (3.8), obviously has all the properties I)—V), with the possible exception of property III). We shall prove that it has this property also. Let ξ and η be arbitrary vectors from $\hat{c}$ and let $\zeta = \xi + \eta$. It is easily seen that

$$\sum_{j=1}^{n} \zeta_j^* \leqq \sum_{j=1}^{n} (\xi_j^* + \eta_j^*) \qquad (n = 1, 2, \cdots).$$

Consequently, according to the condition V')

$$\Phi(\xi + \eta) = \Phi(\zeta^*) \leqq \Phi(\xi^* + \eta^*). \tag{3.10}$$

On the other hand, according to III')

$$\Phi(\xi^* + \eta^*) \leqq \Phi(\xi^*) + \Phi(\eta^*). \tag{3.11}$$

These last two relations imply property III). The lemma is proved.

3. The simplest example of an s. n. function is the function $\Phi_\infty(\xi)$, defined on the cone $\hat{k}$ by

$$\Phi_\infty(\xi) = \xi_1 \qquad (\xi \in \hat{k}).$$

Another simple example of an s. n. function is the function $\Phi_1(\xi)$, for which

$$\Phi_1(\xi) = \sum_j \xi_j \qquad (\xi \in \hat{k}).$$

It is obvious that

$$\Phi_\infty(\xi) = \max_j |\xi_j| \quad \text{and} \quad \Phi_1(\xi) = \sum_j |\xi_j| \qquad (\xi \in \hat{c}).$$

It turns out that the functions $\Phi_\infty(\xi)$ and $\Phi_1(\xi)$ are the extremal s. n. functions.

2. *For any s.n.function $\Phi(\xi)$ the following inequalities hold*:

$$\xi_1 \leqq \Phi(\xi) \leqq \sum_j \xi_j \qquad (\xi \in \hat{k}). \tag{3.12}$$

In fact, by Lemma 3.1, for any vector

$$\xi = \{\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots\} \in \hat{k}$$

we have

$$\Phi(\xi_1, 0, 0, \cdots) \leqq \Phi(\xi) \leqq \Phi\left(\sum_j \xi_j, 0, 0, \cdots\right),$$

and consequently the inequality (3.12) holds.

Thus it is natural to call the functions $\Phi_\infty(\xi)$ and $\Phi_1(\xi)$, respectively, the *minimal* and *maximal* s. n. functions.

3. *Any s.n.function $\Phi(\xi)$ is continuous.*

This property follows at once from the relations

$$|\Phi(\xi) - \Phi(\eta)| \leqq \Phi(\xi - \eta) \leqq \sum_j |\xi_j - \eta_j| \qquad (\xi, \eta \in \hat{c}).$$

4. We shall say that two s. n. functions $\Phi(\xi)$ and $\Psi(\xi)$ are *equivalent* if

$$\sup_{\xi \in c} \frac{\Phi(\xi)}{\Psi(\xi)} < \infty \quad \text{and} \quad \sup_{\xi \in c} \frac{\Psi(\xi)}{\Phi(\xi)} < \infty.$$

4. *For any s.n.function $\Phi(\xi)$ one has the relation*

$$\sup_{\xi \in \hat{k}} \frac{\Phi(\xi)}{\xi_1} = \sup_n \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots). \tag{3.13}$$

In particular, in order that the s. n. function $\Phi(\xi)$ be equivalent to the minimal one, it is necessary and sufficient that

$$\sup_n \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) < \infty. \tag{3.14}$$

In fact, on the one hand

$$\sup_n \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) \leqq \sup_{\xi \in \hat{k}} \frac{\Phi(\xi)}{\xi_1}. \tag{3.15}$$

On the other hand, for any vector $\xi = \{\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots\} \in \hat{k}$ we have

$$\frac{\Phi(\xi)}{\xi_1} = \Phi\left(1, \frac{\xi_2}{\xi_1}, \cdots, \frac{\xi_n}{\xi_1}, 0, 0, \cdots\right) \leqq \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots).$$

Thus

$$\sup_{\xi \in \hat{k}} \frac{\Phi(\xi)}{\xi_1} \leqq \sup_n \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots). \tag{3.16}$$

From (3.15) and (3.16) follows (3.13).

Since for any s. n. function $\Phi(\xi)$ one has

$$\frac{\Phi(\xi)}{\xi_1} \geqq 1 \qquad (\xi \in \hat{k}),$$

this function is equivalent to the minimal one if and only if

$$\sup_{\xi \in \hat{k}} \frac{\Phi(\xi)}{\xi_1} < \infty.$$

This last condition coincides, by virtue of (3.13), with the condition (3.14)

5. (S. T. Kuroda [1]). *For any s.n. function $\Phi(\xi)$ one has*

$$\sup_{\xi \in \hat{k}} \left\{ \frac{1}{\Phi(\xi)} \sum_j \xi_j \right\} = \sup_n \frac{n}{\Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots)}. \tag{3.17}$$

In particular, the s. n. function $\Phi(\xi)$ is equivalent to the maximal one if and only if

$$\sup_n \frac{n}{\Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots)} < \infty. \tag{3.18}$$

Obviously,

$$\sup_n \frac{n}{\Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots)} \leqq \sup_{\xi \in \hat{k}} \left\{ \frac{1}{\Phi(\xi)} \sum_j \xi_j \right\}. \tag{3.19}$$

Let $\xi = \{\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots\}$ be an arbitrary vector from $\hat{k}$, and

$\eta = \{\eta_j\}$ the vector defined by

$$\eta_1 = \eta_2 = \cdots = \eta_n = \frac{1}{n}\sum_j \xi_j, \quad \eta_{n+j} = 0 \qquad (j = 1, 2, \cdots).$$

Then it is easily seen that

$$\sum_{j=1}^{p} \eta_j \leqq \sum_{j=1}^{p} \xi_j \qquad (p = 1, 2, \cdots),$$

and consequently

$$\Phi(\eta) = \left\{\frac{1}{n}\sum_j \xi_j\right\} \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) \leqq \Phi(\xi)$$

and

$$\frac{1}{\Phi(\xi)}\sum_j \xi_j \leqq \frac{n}{\Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots)}. \tag{3.20}$$

The relation (3.17) follows from (3.19) and (3.20).

Since for any s.n. function $\Phi(\xi)$ one has

$$\frac{1}{\Phi(\xi)}\sum_j \xi_j \geqq 1 \qquad (\xi \in \hat{k}),$$

by virtue of (3.17) the condition (3.18) is necessary and sufficient for the equivalence of the function $\Phi(\xi)$ and the maximal s. n. function.

5. In this subsection we shall clarify the relationship between s. n. functions and invariant norms on the ideal $\mathfrak{R}$ of all finite-dimensional operators.

THEOREM 3.1. *A one-to-one correspondence holds between s.n. functions $\Phi(\xi)$ on $\hat{k}$ and invariant norms $|A|_{\mathfrak{S}}$ on $\mathfrak{R}$. Namely:*

Let $|A|_{\mathfrak{S}}$ be any invariant norm on $\mathfrak{R}$. Then the equality

$$\Phi(s(A)) = |A|_{\mathfrak{S}} \quad (A \in \mathfrak{R};\ s(A) = \{s_j(A)\}) \tag{3.21}$$

defines an s.n. function $\Phi(\xi)$. Conversely, if $\Phi(\xi)$ is any s.n. function, then the equality

$$|A|_{\Phi} = \Phi(s(A)) \qquad (A \in \mathfrak{R}) \tag{3.22}$$

defines an invariant norm on $\mathfrak{R}$.

PROOF. The definition (3.21) of the function $\Phi(\xi)$ can be replaced by the equivalent definition

$$\Phi(\xi) = \Big|\sum_j \xi_j(\cdot, \phi_j)\phi_j\Big|_{\mathfrak{S}} \qquad (\xi = \{\xi_j\} \in \hat{k}),$$

where $\{\phi_j\}_1^\infty$ is any fixed orthonormal basis of the space $\mathfrak{H}$. It is easily deduced from the properties of an invariant norm that the function $\Phi(\xi)$ satisfies the conditions I)—V), and is thus an s. n. function.

Conversely, if $\Phi(\xi)$ is some s. n. function, then (3.22) defines a functional which obviously has the properties 1) and 2) of a symmetric norm. This functional also has the property 3). In fact, if $A, B \in \mathfrak{R}$, then according to Lemma II.4.2

$$\sum_{j=1}^{n} s_j(A + B) \leqq \sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B),$$

and consequently by the property V') of an s. n. function,

$$|A + B|_\Phi = \Phi(s(A + B)) \leqq \Phi(s(A) + s(B)).$$

Since

$$\Phi(s(A) + s(B)) \leqq \Phi(s(A)) + \Phi(s(B)),$$

it follows that

$$|A + B|_\Phi \leqq |A|_\Phi + |B|_\Phi.$$

The condition 4') is likewise satisfied by the functional $|A|_\Phi$, since for any unitary operators U and V

$$s_j(VAU) = s_j(A) \qquad (j = 1, 2, \cdots; A \in \mathfrak{R}).$$

Finally, for any one-dimensional operator A

$$s_1(A) = |A|, \quad s_{j+1}(A) = 0 \qquad (j = 1, 2, \cdots),$$

and so

$$|A|_\Phi = \Phi(|A|, 0, 0, \cdots) = |A|.$$

The theorem is proved.

COROLLARY 3.1. *Every invariant norm on the ideal* $\mathfrak{R}$ *is a symmetric norm on this ideal.*

In fact, let $|A|_\mathfrak{S}$ be an invariant norm on $\mathfrak{R}$, and let $\Phi(\xi)$ be the s. n. function generated by this norm. Since for any operators $B, C \in \mathfrak{R}$ one has

$$s_j(BAC) \leqq |B|\,|C|\,s_j(A) \qquad (A \in \mathfrak{R};\, j = 1, 2, \cdots),$$

it follows from the properties of an s. n. function that

$$\Phi(s(BAC)) \leqq \Phi(|B|\,|C|\,s(A)) = |B|\,|C|\,\Phi(s(A)),$$

i.e.

$$|BAC|_\mathfrak{S} \leqq |B|\,|C|\,|A|_\mathfrak{S}.$$

§ 4. Symmetrically-normed ideals generated by a symmetric norming function

1. Let $\xi = \{\xi_j\}$ be an arbitrary sequence of real numbers, and let

$$\xi^{(n)} = \{\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots\}.$$

Then for any s.n. function $\Phi(\xi)$ the sequence $\{\Phi(\xi^{(n)})\}$ is nondecreasing.

Let us denote by $\boldsymbol{c}_\Phi$ the set of all vectors $\xi \in \boldsymbol{c}_0$ for which

$$\sup_n \Phi(\xi^{(n)}) < \infty .$$

We extend the domain of the function $\Phi(\xi)$ by putting, for each $\xi \in \boldsymbol{c}_\Phi$,

$$\Phi(\xi) = \lim_{n\to\infty} \Phi(\xi^{(n)}).$$

From the definition of the set $\boldsymbol{c}_\Phi$ and the properties of an s. n. function $\Phi(\xi)$ it easily follows that the set $\boldsymbol{c}_\Phi$ has the following properties:

a) if $\xi, \eta \in \boldsymbol{c}_\Phi$, then $\xi + \eta \in \boldsymbol{c}_\Phi$;

b) if α is any real number and $\xi \in \boldsymbol{c}_\Phi$, then $\alpha\xi \in \boldsymbol{c}_\Phi$;

c) if $\xi = \{\xi_j\} \in \boldsymbol{c}_\Phi$ and if the vector $\eta = \{\eta_j\} \in \boldsymbol{c}_0$ satisfies the condition

$$\sum_{j=1}^{n} \eta_j^* \leq \sum_{j=1}^{n} \xi_j^* \qquad (n = 1, 2, \cdots),$$

then η also belongs to $\boldsymbol{c}_\Phi$.

From the last property follows the property

d) a vector $\xi = \{\xi_j\}$ belongs to $\boldsymbol{c}_\Phi$ if and only if $\xi^* = \{\xi_j^*\} \in \boldsymbol{c}_\Phi$.

It is easily seen that the s. n. function $\Phi(\xi)$ preserves its properties I)—V), or, what is the same, the properties I′)—V′), in the extended domain $\boldsymbol{c}_\Phi$.

We shall call the lineal $\boldsymbol{c}_\Phi$ the *natural domain* of the s. n. function $\Phi(\xi)$.

2. With every s. n. function $\Phi(\xi)$ we associate the set $\mathfrak{S}_\Phi$ of all operators $X \in \mathfrak{S}_\infty$ for which $s(X) = \{s_j(X)\} \in \boldsymbol{c}_\Phi$, and for every $X \in \mathfrak{S}_\Phi$ we put

$$|X|_\Phi = \Phi(s(X)). \tag{4.1}$$

It is obvious that $X \in \mathfrak{S}_\Phi$ if and only if

$$\sup_n |X_n|_\Phi < \infty,$$

where X_n is the nth partial Schmidt series of the operator X, and that

$$|X|_\Phi = \lim_{n\to\infty} |X_n|_\Phi = \Phi(s(X)).$$

THEOREM 4.1. *Let $\Phi(\xi)$ be any s. n. function. Then the set $\mathfrak{S}_\Phi$ is an s. n.*

ideal with the norm

$$|A|_\Phi = |A|_{\mathfrak{S}_\Phi} = \Phi(s(A)) \qquad (A \in \mathfrak{S}_\Phi). \tag{4.2}$$

PROOF. Let $A_1, A_2 \in \mathfrak{S}_\Phi$; then $s(A_1) + s(A_2) \in \boldsymbol{c}_\Phi$. According to Theorem II.4.2

$$\sum_{j=1}^{n} s_j(A_1 + A_2) \leqq \sum_{j=1}^{n} s_j(A_1) + \sum_{j=1}^{n} s_j(A_2) \qquad (n = 1, 2, \cdots)$$

and consequently $s(A_1 + A_2) \in \boldsymbol{c}_\Phi$, i.e. the operator $A_1 + A_2$ belongs to $\mathfrak{S}_\Phi$, and by property V′),

$$|A_1 + A_2|_\Phi \leqq |A_1|_\Phi + |A_2|_\Phi.$$

It is easily seen that for any operator $A \in \mathfrak{S}_\Phi$ and any complex number λ one has $\lambda A \in \mathfrak{S}_\Phi$ and

$$|\lambda A|_\Phi = |\lambda|\; |A|_\Phi.$$

The space $\mathfrak{S}_\Phi$ is complete. In fact, let $\{A_n\}_1^\infty$ be a Cauchy sequence of operators in $\mathfrak{S}_\Phi$. Then by virtue of the left side of the bound (3.12) we have

$$|A_m - A_n| = s_1(A_m - A_n) \leqq |A_m - A_n|_\Phi,$$

and so the sequence of operators $\{A_n\}_1^\infty$ converges in the uniform norm to some completely continuous operator A. For this operator

$$\lim_{n\to\infty} s_j(A_n) = s_j(A) \qquad (j = 1, 2, \cdots).$$

We have

$$\begin{aligned} &\Phi(s_1(A_r), s_2(A_r), \cdots, s_n(A_r), 0, 0, \cdots) \\ &\qquad\qquad \leqq \sup_p |A_p|_\Phi (< \infty) \qquad (r = 1, 2, \cdots). \end{aligned} \tag{4.3}$$

Since the function $\Phi(\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots)$ is continuous, then, passing to the limit $r \to \infty$, we obtain

$$\Phi(s_1(A), s_2(A), \cdots, s_n(A), 0, 0, \cdots) \leqq \sup_p |A_p|_\Phi \qquad (n = 1, 2, \cdots).$$

Thus $A \in \mathfrak{S}_\Phi$.

We shall show that the sequence of operators $\{A_n\}_1^\infty$ tends to the operator A in the norm of the space $\mathfrak{S}_\Phi$. Let ϵ be any positive number and let N be such that for all $p, q > N$

$$|A_p - A_q|_\Phi < \epsilon.$$

Then

$$\Phi(s_1(A_p - A_q), s_2(A_p - A_q), \cdots, s_n(A_p - A_q), 0, 0, \cdots) < \epsilon \tag{4.4}$$
$$(n = 1, 2, \cdots; p, q > N).$$

Bearing in mind that

$$\lim_{q \to \infty} s_j(A_p - A_q) = s_j(A_p - A) \qquad (j = 1, 2, \cdots),$$

and letting $q \to \infty$ in (4.4), we find that

$$\Phi(s_1(A_p - A), \cdots, s_n(A_p - A), 0, 0, \cdots) \leqq \epsilon \qquad (n = 1, 2, \cdots; p > N).$$

It follows that

$$|A_p - A|_\Phi \leqq \epsilon \qquad (p > N).$$

To conclude the proof of the theorem, it remains to prove property 4) of a symmetric norm. This property follows from the earlier-proved inequalities

$$s_j(BAC) \leqq |B|\,|C|\, s_j(A) \qquad (j = 1, 2, \cdots;\ A \in \mathfrak{S}_\Phi;\ B, C \in \mathfrak{R}).$$

The theorem is proved.

Let us note a further important property of the norm $|A|_\Phi$.

DOMINANCE PROPERTY. *If $A \in \mathfrak{S}_\Phi$ and the operator B ($\in \mathfrak{S}_\infty$) has the property that*

$$\sum_{j=1}^{n} s_j(B) \leqq \sum_{j=1}^{n} s_j(A) \qquad (n = 1, 2, \cdots),$$

then $B \in \mathfrak{S}_\Phi$ and $|B|_\Phi \leqq |A|_\Phi$.

Theorem 4.1 can be supplemented by the following result.

1. *In order that the s.n. ideals $\mathfrak{S}_{\Phi_1}$ and $\mathfrak{S}_{\Phi_2}$ coincide elementwise, it is necessary and sufficient that the s.n. functions $\Phi_1(\xi)$ and $\Phi_2(\xi)$ be equivalent.*

The sufficiency of this condition is obvious, and its necessity follows at once from Theorem 2.1 on the topological equivalence of norms.

3. We present some further results concerning the spaces $\mathfrak{S}_\Phi$, which follow from various theorems proved earlier.

THEOREM 4.2. *Let A be an operator from $\mathfrak{S}_\Phi$ and let $\{P_j\}_1^\omega$ ($\omega \leqq \infty$) be a system of mutually orthogonal orthoprojectors. Then the operator*

$$\hat{A} = \sum_{j=1}^{\omega} P_j A P_j$$

also belongs to the space $\mathfrak{S}_\Phi$, and

$$|\hat{A}|_\Phi \leqq |A|_\Phi.$$

By virtue of property V′) of s. n. functions this theorem follows at once from Theorem II.5.1.

Let $\{\phi_j\}_1^\omega$ $(\omega \leqq \infty)$ be some orthonormal system, and P_j $(j = 1, 2, \cdots, \omega)$ the operator of orthogonal projection onto the subspace generated by the unit vector ϕ_j $(j = 1, 2, \cdots, \omega)$. Then the operator $\hat{A}$ has the form

$$\hat{A} = \sum_{j=1}^{\omega} (A\phi_j, \phi_j)(\cdot, \phi_j)\phi_j.$$

Let us assume that the vectors of the sequence $\{\phi_j\}$ are numbered so that the sequence $\{|(A\phi_j, \phi_j)|\}_1^\omega$ is nonincreasing. Then obviously $s_j(\hat{A}) = |(A\phi_j, \phi_j)|$ $(j = 1, 2, \cdots)$.

It follows from Theorem 4.2 that $\hat{A} \in \mathfrak{S}_\Phi$ and

$$\Phi(|(A\phi_1, \phi_1)|, |(A\phi_2, \phi_2)|, \cdots) \leqq |A|_\Phi. \tag{4.5}$$

THEOREM 4.3. *If $A \in \mathfrak{S}_\Phi$ then the sequence of its eigenvalues $\{\lambda_j(A)\}$ satisfies the relation*

$$\Phi(|\lambda_1(A)|, |\lambda_2(A)|, \cdots) \leqq |A|_\Phi \qquad (A \in \mathfrak{S}_\Phi). \tag{4.6}$$

PROOF. According to Lemma I.4.1, an orthonormal system $\{\phi_j\}_1^{\nu(A)}$ can be chosen so that

$$(A\phi_j, \phi_j) = \lambda_j(A) \qquad (j = 1, 2, \cdots, \nu(A)).$$

Following this, the relation (4.5) assumes the form (4.6).

THEOREM 4.4. *Let A be a bounded linear operator with imaginary component $A_{\mathscr{J}} \in \mathfrak{S}_\Phi$ and let λ_j $(j = 1, 2, \cdots, \nu_{\mathscr{J}}(A))$ be the system of all nonreal eigenvalues of the operator A, arbitrarily enumerated in decreasing order of the absolute value of their imaginary parts, taking their algebraic multiplicities into account. Then*

$$\Phi(|\operatorname{Im}\lambda_1|, |\operatorname{Im}\lambda_2|, \cdots) \leqq |A_{\mathscr{J}}|_\Phi. \tag{4.7}$$

This theorem is a consequence of Lemma II.6.1 and property V′) of the function $\Phi(\xi)$.

§5. A criterion for an operator to belong to an s.n. ideal $\mathfrak{S}_\Phi$.

1. We shall present below a criterion for a given operator to belong to a space $\mathfrak{S}_\Phi$, which is used, in particular, to construct the conjugate spaces of the spaces $\mathfrak{S}_\Phi$. For the proof of this criterion we need two lemmas.

LEMMA 5.1. *Suppose that the bounded linear operator A is the weak limit of a sequence of operators $\{A_n\}_1^\infty$ from $\mathfrak{S}_\infty$, i.e. for any $f, g \in \mathfrak{H}$*

$$\lim_{n \to \infty} (A_n f, g) = (Af, g).$$

If

(5.1) $$\lim_{j \to \infty} \sup_m s_j(A_m) = 0,$$

then the operator A is completely continuous and[4)]

(5.2) $$\sum_{j=1}^{n} s_j(A) \leqq \sum_{j=1}^{n} \sup_m s_j(A_m) \qquad (n = 1, 2, \cdots).$$

PROOF. Let the Schmidt expansion of the operator A_n have the form

$$A_n = \sum_{j=1}^{\infty} s_j(A_n) (\cdot, \phi_j^{(n)}) \psi_j^{(n)} \qquad (n = 1, 2, \cdots)$$

and let

$$T_{n,k} = \sum_{j=1}^{k} s_j(A_n) (\cdot, \phi_j^{(n)}) \psi_j^{(n)}$$

be the partial sums of this expansion.

We choose a sequence of indices n_r $(r = 1, 2, \cdots)$ such that the sequences of numbers $\{s_j(A_{n_r})\}_{r=1}^{\infty}$ converge, and the sequences of vectors $\{\phi_j^{(n_r)}\}_{r=1}^{\infty}$ and $\{\psi_j^{(n_r)}\}_{r=1}^{\infty}$ $(j = 1, 2, \cdots, k)$ converge weakly. We denote the limits of these sequences by s_j, ϕ_j and ψ_j respectively. Then the operator

$$T_k = \sum_{j=1}^{k} s_j (\cdot, \phi_j) \psi_j$$

is obviously the weak limit of the sequence of operators $T_{n_r,k}$ $(r = 1, 2, \cdots)$.

Since

$$|A_n - T_{n,k}| \leqq s_{k+1}(A_n) \leqq \sup_n s_{k+1}(A_n),$$

it follows that

$$|(A_{n_r} f, \chi) - (T_{n_r,k} f, \chi)| \leqq \sup_n s_k(A_n) \qquad (|f| = |\chi| = 1).$$

Passing to the limit $r \to \infty$, we obtain

$$|(Af, \chi) - (T_k f, \chi)| \leqq \sup_n s_k(A_n) \qquad (|f| = |\chi| = 1),$$

or

[4)] The relations (5.2) remain valid if we replace the numbers $\sup_m s_j(A_m)$ in them by the numbers $\overline{\lim}_{m \to \infty} s_j(A_m)$.

$$|A - T_k| \leqq \sup_n s_k(A_n).$$

It follows from the hypothesis (5.1) that

$$\lim_{k\to\infty} |A - T_k| = 0.$$

Consequently the operator A is completely continuous.

We choose a unitary operator U and an orthonormal system $\{\chi_j\}_1^n$ such that

$$(UA\chi_j, \chi_j) = s_j(A) \qquad (j = 1, 2, \cdots, n).$$

According to Lemma II.4.1,

$$\left|\sum_{j=1}^n (UA_{n_r}\chi_j, \chi_j)\right| \leqq \sum_{j=1}^n s_j(A_{n_r}),$$

and so

$$\left|\sum_{j=1}^n (UA_{n_r}\chi_j, \chi_j)\right| \leqq \sum_{j=1}^n \sup_m s_j(A_m).$$

Passing to the limit $n_r \to \infty$, we have

$$\sum_{j=1}^n s_j(A) = \sum_{j=1}^n (UA\chi_j, \chi_j) \leqq \sum_{j=1}^n \sup_m s_j(A_m).$$

The lemma is proved.

2. Among the spaces $\mathfrak{S}_\Phi$ there are spaces consisting of all completely continuous operators. Such will be, for example, the space $\mathfrak{S}_F$ generated by any one of the functions

$$F_n(\xi) = \sum_{j=1}^n \xi_j^* \qquad (\xi = \{\xi_j\} \in \mathbf{c}_0),$$

where n is an arbitrary fixed integer.

The following lemma enables us to distinguish these spaces.

LEMMA 5.2. *Let $\Phi(\xi)$ be an arbitrary s.n.function. In order that the sets $\mathfrak{S}_\Phi$ and $\mathfrak{S}_\infty$ coincide, it is necessary and sufficient that the function $\Phi(\xi)$ be equivalent to the minimal s.n.function, i.e.*

$$\sup_n \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) < \infty. \tag{5.3}$$

This follows at once from results 1 of §4 and 4 of §3.

THEOREM 5.1. *Let $\Phi(\xi)$ be an arbitrary s.n.function not equivalent to the minimal one (i.e. $\mathfrak{S}_\Phi$ and $\mathfrak{S}_\infty$ do not coincide elementwise). If an opera-*

tor $A \in \mathfrak{R}$ is the weak limit of a sequence of operators $\{A_m\}_1^\infty$ from $\mathfrak{S}_\Phi$ and if

$$\sup_m |A_m|_\Phi < \infty,$$

then the operator A also belongs to $\mathfrak{S}_\Phi$, and[5)]

$$|A|_\Phi \leqq \sup_m |A_m|_\Phi. \tag{5.4}$$

PROOF. Since

$$\begin{aligned} &s_n(A_m)\,\Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots) \\ &\leqq \Phi(s_1(A_m),\cdots,s_n(A_m),0,0,\cdots) \leqq M \qquad (m,n=1,2,\cdots), \end{aligned} \tag{5.5}$$

where $M = \sup_m |A_m|_\Phi$, we have

$$s_n(A_m) \leqq \frac{M}{\Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)} \qquad (n,m=1,2,\cdots).$$

Consequently, by the hypotheses of the theorem and Lemma 5.2,

$$\lim_{n\to\infty} \sup_m s_n(A_m) = 0.$$

It follows, by Lemma 5.1, that the operator A is completely continuous.

We choose a sequence of indices n_r $(r = 1,2,\cdots)$ such that the sequences of numbers $\{s_j(A_{n_r})\}_{r=1}^\infty$, for $j = 1,2,\cdots,k$, converge. It follows from the relation (5.5) that

$$\Phi(s_1,s_2,\cdots,s_k,0,0,\cdots) \leqq M \qquad (k=1,2,\cdots),$$

where

$$s_j = \lim_{r\to\infty} s_j(A_{n_r}) \qquad (j=1,2,\cdots,k).$$

By the same Lemma 5.1 (see the footnote to it)

$$\sum_{j=1}^r s_j(A) \leqq \sum_{j=1}^r s_j \qquad (r=1,2,\cdots,k).$$

Consequently

$$\Phi(s_1(A),s_2(A),\cdots,s_k(A),0,0,\cdots) \leqq M \qquad (k=1,2,\cdots).$$

This means that $A \in \mathfrak{S}_\Phi$. The theorem is proved.

[5)] The relation (5.4) remains valid if the number $\sup |A_m|_\Phi$ in it is replaced by the number $\overline{\lim}\, |A_m|_\Phi$.

The following criterion for an operator A to belong to an ideal $\mathfrak{S}_\Phi$ is a simple consequence of the theorem just proved.

THEOREM 5.2. *Let $\mathfrak{S}_\Phi$ be an s.n. ideal, not coinciding elementwise with $\mathfrak{S}_\infty$. Further, let P_n $(n = 1, 2, \cdots)$ be a monotonically increasing sequence of finite-dimensional orthoprojectors which tends strongly to the identity operator*:

$$\lim_{n \to \infty} |P_n f - f| = 0 \qquad (f \in \mathfrak{H}). \tag{5.6}$$

Then $A \in \mathfrak{S}_\Phi$ if and only if

$$M = \sup_n |P_n A P_n|_\Phi < \infty. \tag{5.7}$$

REMARK 5.1. For an s.n. ideal $\mathfrak{S}_\Phi$ which coincides elementwise with $\mathfrak{S}_\infty$ Theorem 5.2 does not hold, since the condition (5.7) is in this case fulfilled for all bounded operators.

§6. Separable s.n. ideals

1. Let us denote by $\mathfrak{K}_n$ $(n = 1, 2, \cdots)$ the set of all finite-dimensional operators of dimension $\leqq n$. Obviously $\mathfrak{K} = \bigcup_n \mathfrak{K}_n$.

LEMMA 6.1. *Let $\Phi(\xi)$ be an s.n. function and $A \in \mathfrak{S}_\Phi$. Then*

$$\min_{K \in \mathfrak{K}_n} |A - K|_\Phi = |A - A_n|_\Phi = \Phi(s_{n+1}(A), s_{n+2}(A), \cdots) \tag{6.1}$$

and

$$\inf_{K \in \mathfrak{K}} |A - K|_\Phi = \lim_{n \to \infty} \Phi(s_{n+1}(A), s_{n+2}(A), \cdots), \tag{6.2}$$

where A_n $(n = 1, 2, \cdots)$ is the nth partial Schmidt series of the operator A.

PROOF. In fact, according to Corollary II.2.1

$$s_j(A - K) \geqq s_{n+j}(A) \qquad (K \in \mathfrak{K}_n;\ n, j = 1, 2, \cdots),$$

and consequently by result 2 of §2

$$|A - K|_\Phi \geqq |A - A_n|_\Phi.$$

Since equality holds here for $K = A_n$, the relation (6.1) is proved. Since (6.2) follows at once from (6.1), the lemma is proved.

In §14 examples will be given of s.n. ideals in which the set of finite-dimensional operators is not dense. This circumstance suggests the necessity of introducing the subspace $\mathfrak{S}_\Phi^{(0)}$, the closure in the norm of $\mathfrak{S}_\Phi$ of the set $\mathfrak{K}$. By Lemma 6.1, this subspace consists of all operators

$A \in \mathfrak{S}_\Phi$ for which

$$\lim_{n\to\infty} \Phi(s_{n+1}(A), s_{n+2}(A), \cdots) = 0$$

or

$$\lim_{n,p\to\infty} \Phi(s_{n+1}(A), s_{n+2}(A), \cdots, s_{n+p}(A), 0, 0, \cdots) = 0.$$

We shall say that an s.n. function $\Phi(\xi)$ is *mononormalizing* if the condition

$$\lim_{n\to\infty} \Phi(\xi_{n+1}, \xi_{n+2}, \cdots) = 0$$

or the equivalent condition

$$\lim_{n,p\to\infty} \Phi(\xi_{n+1}, \xi_{n+2}, \cdots, \xi_{n+p}, 0, 0, \cdots) = 0$$

is fulfilled for every $\xi \in c_\Phi$.

By virtue of what was said above, the spaces $\mathfrak{S}_\Phi$ and $\mathfrak{S}_\Phi^{(0)}$ coincide if and only if $\Phi(\xi)$ is a mononormalizing function.

It is easily seen that the minimal and maximal s.n. functions are mononormalizing.

It follows from Lemma 6.1 that for a mononormalizing function Φ and for any operator A from $\mathfrak{S}_\Phi$, the Schmidt series of A converges to A in the norm $|\ |_\Phi$.

Every s.n. function which is not mononormalizing will be referred to as *binormalizing*.

It is not difficult to prove the following theorem.

THEOREM 6.1. *The subspace $\mathfrak{S}_\Phi^{(0)}$ is a separable s.n. ideal of the ring $\mathfrak{R}$.*

PROOF. We shall first show that, together with an operator $A \in \mathfrak{S}_\Phi^{(0)}$, every operator BAC $(B, C \in \mathfrak{R})$ also belongs to $\mathfrak{S}_\Phi^{(0)}$.

Let A_n be the nth partial Schmidt series of the operator A. The operators BA_nC are finite-dimensional and consequently belong to $\mathfrak{S}_\Phi^{(0)}$. Since

$$|BA_nC - BAC|_\Phi \leqq |B|\,|C|\,|A_n - A|_\Phi,$$

and $A \in \mathfrak{S}_\Phi^{(0)}$, by Lemma 6.1 we have

$$\lim_{n\to\infty} |BA_nC - BAC|_\Phi = 0.$$

Thus $BAC \in \mathfrak{S}_\Phi^{(0)}$.

Let $\mathfrak{N}$ be some countable set of vectors, dense in $\mathfrak{H}$, and $\mathfrak{M}$ the set of all finite-dimensional operators of the form

$$K = \sum_j (\cdot, \chi_j)\omega_j,$$

where $\chi_j, \omega_j \in \mathfrak{N}$. Obviously $\mathfrak{M}$ is also a countable set. To prove that $\mathfrak{S}_\Phi^{(0)}$ is separable it is sufficient to show that $\mathfrak{M}$ is dense in $\mathfrak{K}$ in the metric generated by the norm of $\mathfrak{S}_\Phi$, and this is obvious. The theorem is proved.

THEOREM 6.2. *Every separable s.n. ideal coincides with some ideal* $\mathfrak{S}_\Phi^{(0)}$.

PROOF. Let $\mathfrak{S}$ be any separable s.n. ideal with norm $|A|_\mathfrak{S}$. The norm $|A|_\mathfrak{S}$ generates an invariant norm on the set $\mathfrak{K}$ $(\subset \mathfrak{S})$ of all finite-dimensional operators. Hence according to Theorem 3.1 we can find an s.n. function $\Phi(\xi)$ such that

$$|A|_\mathfrak{S} = \Phi(s(A)) = |A|_\Phi \qquad (A \in \mathfrak{K}).$$

We shall show that $\mathfrak{S} = \mathfrak{S}_\Phi^{(0)}$. Let us assume the contrary, i.e. that $\mathfrak{S}$ contains an operator $B \notin \mathfrak{S}_\Phi^{(0)}$. Since for any finite-dimensional orthoprojector P

$$|PBP|_\Phi = |PBP|_\mathfrak{S} \leqq |B|_\mathfrak{S},$$

according to Theorem 5.2 $B \in \mathfrak{S}_\Phi$.

Since $B \notin \mathfrak{S}_\Phi^{(0)}$, there exists $\delta > 0$ such that

$$|B - B_n|_\Phi > \delta \qquad (n = 1, 2, \cdots), \tag{6.3}$$

where B_n is the nth partial Schmidt series of the operator B:

$$B = \sum_{j=1}^\infty s_j(B)(\cdot, \phi_j)\psi_j.$$

From (6.3) it follows that for any positive integer n one can find an integer $m > n$ such that

$$\left| \sum_{j=n}^m s_j(B)(\cdot, \phi_j)\psi_j \right|_\Phi > \delta,$$

and this implies the existence of an increasing sequence of positive integers $\{n_k\}_{k=1}^\infty$ $(n_1 = 1)$ for which the norms of the operators

$$C_k = \sum_{j=n_k}^{n_{k+1}-1} s_j(B)(\cdot, \phi_j)\psi_j$$

exceed δ:

$$|C_k|_\Phi > \delta \qquad (k = 1, 2, \cdots).$$

With every sequence $\alpha = \{\alpha_j\}_1^\infty$ consisting only of zeros and ones we

associate the operator

$$A_\alpha = \sum_{j=1}^{\infty} \alpha_j C_j. \tag{6.4}$$

Obviously $s_j(A_\alpha) \leqq s_j(B)$ $(j = 1, 2, \cdots)$, and so the operator A_α belongs to the space $\mathfrak{S}$. The cardinality of the set of all sequences $\alpha = \{\alpha_j\}_1^\infty$ $(\alpha_j = 0, 1)$ equals the cardinality of the continuum. Since to distinct sequences α' and α'' there correspond distinct operators $A_{\alpha'}$ and $A_{\alpha''}$, the set $\mathfrak{A}$ of all operators of the form (6.4) also has the cardinality of the continuum.

Let $A_{\alpha'}$ and $A_{\alpha''}$ be two arbitrary distinct operators from $\mathfrak{A}$. Then the operator $A_{\alpha'} - A_{\alpha''}$ has the form

$$A_{\alpha'} - A_{\alpha''} = \sum_{j=1}^{\infty} \epsilon_j C_j,$$

where not all of the numbers ϵ_j equal zero, and each of them assumes one of the values $-1, 0, 1$. If for some integer r the number ϵ_r equals $+1$ or -1, then, as is easily seen,

$$s_j(A_{\alpha'} - A_{\alpha''}) \geqq s_j(C_r).$$

Therefore if $\alpha' \neq \alpha''$, then for some r we will have

$$|A_{\alpha'} - A_{\alpha''}|_{\mathfrak{S}} \geqq |C_r|_{\mathfrak{S}} = |C_r|_\Phi > \delta.$$

Since the set $\mathfrak{A}$ has the cardinality of the continuum, we conclude that it, and together with it the space $\mathfrak{S}$, is nonseparable. The theorem is proved.

COROLLARY 6.1. *An s.n. ideal $\mathfrak{S}_\Phi$ is nonseparable if and only if the function Φ is binormalizing, i.e., $\mathfrak{S}_\Phi \neq \mathfrak{S}_\Phi^{(0)}$.*

2. THEOREM 6.3. *Let X_n $(n = 1, 2, \cdots)$ be a sequence of selfadjoint operators from $\mathfrak{R}$ which converges strongly to some operator $X \in \mathfrak{R}$. If $\mathfrak{S}$ is a separable s.n. ideal and $A \in \mathfrak{S}$, then the sequences $\{X_nA\}_1^\infty$, $\{AX_n\}_1^\infty$, $\{X_nAX_n\}_1^\infty$ converge in the norm of the ideal $\mathfrak{S}$ to the operators XA, AX and XAX, respectively.*

PROOF. We shall carry out the proof for the first of the indicated sequences. The proof for the other two is analogous.

Let us first consider the case in which A is a finite-dimensional operator:

$$A = \sum_{j=1}^{r} (\cdot, \phi_j)\psi_j \qquad (|\phi_j| = 1).$$

Then

$$X_nA - XA = \sum_{j=1}^{r} (\cdot, \phi_j)(X_n\psi_j - X\psi_j)$$

and so

$$|X_nA - XA|_{\mathfrak{S}} \leqq \sum_{j=1}^{r} |X_n\psi_j - X\psi_j|,$$

which implies that

$$\lim_{n\to\infty} |X_nA - XA|_{\mathfrak{S}} = 0.$$

In the general case we represent the operator A in the form of a sum of a finite-dimensional operator K and an operator L which is small in $\mathfrak{S}$-norm. Then

$$|X_nA - XA|_{\mathfrak{S}} \leqq |X_nK - XK|_{\mathfrak{S}} + (|X_n| + |X|)|L|_{\mathfrak{S}}. \tag{6.5}$$

Since

$$\sup_n |X_n| < \infty,$$

the second term in the right side of (6.5) can be made as small as desired by making $|L|_{\mathfrak{S}}$ sufficiently small, and, as was proved, the first term in the right side of (6.5) tends to zero as $n \to \infty$. The theorem is proved.

§7. The symmetrically-normed ideals $\mathfrak{S}_p$.

1. Let us consider the function

$$\Phi_p(\xi) = \left(\sum_j |\xi_j|^p\right)^{1/p} \qquad (0 < p \leqq \infty; \xi \in \hat{c}). \tag{7.1}$$

For $p = \infty$, the equality (7.1) is given, as usual, the following meaning:

$$\Phi_\infty(\xi) = \max_j |\xi_j|.$$

The function $\Phi_p(\xi)$ obviously satisfies the conditions I), II), IV), V), which define an s.n. function, and for $p \geqq 1$ it also satisfies the condition III).

Thus for $1 \leqq p \leqq \infty$ the function $\Phi_p(\xi)$ is an s.n. function. It is easily seen that its natural domain $\boldsymbol{c}_{\Phi_p}$ coincides with $\boldsymbol{l}_p$, i.e., consists of all sequences $\xi = \{\xi_j\}_1^\infty \in \boldsymbol{c}_0$ for which

$$\sum_j |\xi_j|^p < \infty.$$

It is obvious that for any sequence $\xi = \{\xi_j\}_1^\infty \in \boldsymbol{l}_p$

$$\lim_{n\to\infty}\left(\sum_{j=n+1}^{\infty}|\xi_j|^p\right)^{1/p}=0,$$

and consequently the function $\Phi_p(\xi)$ is mononormalizing for $1 \leqq p < \infty$. For $p = \infty$ the function $\Phi_p(\xi) = \max_j|\xi_j|$, as we have already mentioned, also is mononormalizing.

As a corollary of the preceding theorems we obtain the following result.

THEOREM 7.1. *The collection $\mathfrak{S}_p$ $(1 \leqq p \leqq \infty)$, consisting of all completely continuous operators A for which*

$$\sum_{j=1}^{\infty} s_j^p(A) < \infty, \tag{7.2}$$

is a separable s.n. ideal with the norm

$$|A|_p = |A|_{\Phi_p} = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p}. \tag{7.3}$$

The set of all finite-dimensional operators is dense in $\mathfrak{S}_p$, and

$$\min_{K\in\mathfrak{K}_n}|A-K|_p=\left(\sum_{j=n+1}^{\infty} s_j^p(A)\right)^{1/p} \qquad (A\in\mathfrak{S}_p;\ n=1,2,\cdots).$$

2. In the sequel we shall also maintain the notation $\mathfrak{S}_p$ for the class of all completely continuous operators A for which the condition (7.2) is fulfilled for $0 < p < 1$. Here $|A|_p$ will also be defined by the equality (7.3). We remark that in this case the function $|A|_p$ no longer has the properties of a norm.

Let us note some properties of the classes $\mathfrak{S}_p$.

1. *If $0 < p_1 < p_2 \leqq \infty$ and $A \in \mathfrak{S}_{p_1}$, then $A \in \mathfrak{S}_{p_2}$ and*

$$|A|_{p_2} \leqq |A|_{p_1}. \tag{7.4}$$

This property follows from the fact that the function

$$f(p)=\left(\sum_{j=1}^{\infty} s_j^p\right)^{1/p} \qquad (0<p\leqq\infty, s_j>0)$$

is nonincreasing.

2. *If the operators A_j $(j = 1, 2, \cdots, n)$ belong respectively to the spaces $\mathfrak{S}_{p_j}$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^n p_j^{-1} \leqq 1$, then the operator $A = A_1A_2\cdots A_n$ belongs to the space $\mathfrak{S}_p$, where $p^{-1} = \sum_{j=1}^n p_j^{-1}$, and*

$$|A|_p \leqq |A_1|_{p_1}|A_2|_{p_2}\cdots|A_n|_{p_n}. \tag{7.5}$$

In particular, if $A \in \mathfrak{S}_p$ $(1 \leqq p \leqq \infty)$ and $B \in \mathfrak{S}_q$, where $p^{-1} + q^{-1} = 1$,

then $AB, BA \in \mathfrak{S}_1$ *and*

$$|AB|_1 \leqq |A|_p|B|_q, \qquad |BA|_1 \leqq |A|_p|B|_q.$$

In fact, putting $f(x) = x^p$ in Theorem II.4.2, we obtain

$$|A|_p \leqq \left(\sum_j s_j^p(A_1)s_j^p(A_2)\cdots s_j^p(A_n)\right)^{1/p}. \tag{7.6}$$

On the other hand, according to the generalized Hölder inequality,

$$\left(\sum_j (s_j(A_1)s_j(A_2)\cdots s_j(A_n))^p\right)^{1/p} \leqq \left(\sum_j s_j^{p_1}(A_1)\right)^{1/p_1}\cdots\left(\sum_j s_j^{p_n}(A_n)\right)^{1/p_n}.$$

3. *If* $A \in \mathfrak{S}_p$ $(p > 0)$, *then for any positive integer* r *we have* $A^r \in \mathfrak{S}_{p/r}$, *and*

$$|A^r|_{p/r} \leqq (|A|_p)^r.$$

This result follows from Corollary II.4.2. For the case of a positive integer p and $r = 2$ it was pointed out by K. Fan [1].

3. In this subsection we shall present some sharpenings of the results of §4.3 for the case $\mathfrak{S}_\Phi = \mathfrak{S}_p$.

4. *If* $A \in \mathfrak{S}_p$ $(p > 0)$, *then*

$$\sum_{j=1}^{\nu(A)} |\lambda_j(A)|^p \leqq (|A|_p)^p, \tag{7.7}$$

and equality holds if and only if A *is a normal operator.*

This result is a consequence of Theorem II.3.1.

REMARK 7.1. For $p = 2$ the result 4 was first obtained by I. Schur, who found an elementary proof for it (cf. §9, Theorem 9.1).

Inequality (7.7) can also be obtained in a very elementary way for the case $p = 1$ (cf. the footnote on p. 98).

An immediate consequence of inequality (7.7) for $p = 1$ is

If $B, C \in \mathfrak{S}_2$, *then*

$$\sum_j |\lambda_j(BC)| < \infty. \tag{7.8}$$

In fact, in this case $A = BC \in \mathfrak{S}_1$ (incidentally, we mention that if B and C run through all of $\mathfrak{S}_2$, then, as is easily seen, A runs through all of $\mathfrak{S}_1$).

It is curious that it required the efforts of several mathematicians to establish the inequality (7.8):[6] following the work of T. Lalesco and

[6] And, at the same time, without the refinement that in the right side of (7.8) one can put $|BC|_1$ or at least $|B|_2|C|_2$ $(\geqq |BC|_1)$ in place of ∞.

S. A. Georgiu, it was finally proved (by far from simple methods) by E. Hille and J. D. Tamarkin in the interesting memoir [1].

5. *If* $A \in \mathfrak{S}_p$ $(1 \leqq p < \infty)$ *and* $\{\phi_j\}_1^\omega$ $(\omega \leqq \infty)$ *is some orthonormal system, then*

$$\sum_{j=1}^{\omega} |(A\phi_j, \phi_j)|^p \leqq (|A|_p)^p. \tag{7.9}$$

For $p > 1$ *the equal sign in* (7.9) *will hold if and only if* A *is a normal operator and, moreover,*

$$A = \sum_{j=1}^{\omega} t_j(\cdot, \phi_j)\phi_j, \qquad t_j = (A\phi_j, \phi_j).$$

This theorem can be obtained as a consequence of the assertions of §4.3, Chapter II applied to the function $f(x) = x^p$.

For $p = 1$ the conditions under which equality holds in (7.9) will be clarified in the following section.

Result 5 admits the following generalization.

6. *Let* $A \in \mathfrak{S}_p$ $(1 \leqq p < \infty)$ *and let* $\{P_j\}_1^\omega$ $(\omega \leqq \infty)$ *be a system of mutually orthogonal orthoprojectors. Then*

$$\left|\sum_{j=1}^{\omega} P_j A P_j\right|_p = \left(\sum_{j=1}^{\omega} |P_j A P_j|_p^p\right)^{1/p} \leqq |A|_p,$$

and for $p > 1$ *equality will hold if and only if*

$$A = \sum_{j=1}^{\omega} P_j A P_j.$$

This result is easily obtained from the general Theorem II.5.2.

Let us note yet another result, which is a simple consequence of Theorem II.6.1.

7. *Let* A *be a bounded linear operator with component* $A_{\mathcal{J}} \in \mathfrak{S}_p$ $(1 \leqq p < \infty)$. *Then*

$$\sum_{j=1}^{\nu_{\mathcal{J}}(A)} |\operatorname{Im} \lambda_j|^p \leqq (|A_{\mathcal{J}}|_p)^p. \tag{7.10}$$

For $p > 1$ *equality holds if and only if* A *is a normal operator.*

In §2 Chapter V it will be shown that the equality

$$\sum_{j=1}^{\nu_{\mathcal{J}}(A)} |\operatorname{Im} \lambda_j| = |A_{\mathcal{J}}|_1$$

is a sufficient condition for the completeness of the system of root vectors of the operator A.

If $A \in \mathfrak{S}_p$ $(1 \leqq p < \infty)$ then besides (7.10) we have

$$\sum_{j=1}^{\nu(A)} |\operatorname{Re} \lambda_j(A)|^p \leqq (|A_{\mathscr{R}}|_p)^p. \tag{7.11}$$

We remark that I. Schur's inequality (inequality (7.7) with $p = 2$) is contained as a corollary in the relations (7.10) and (7.11) for $p = 2$.

8. *If* $A \in \mathfrak{S}_p$ $(p > 0)$, *then*

$$s_n(A) = o(n^{-1/p}). \tag{7.12}$$

This follows at once from the relation $\lim_{n\to\infty} na_n = 0$, which holds for every convergent series $\sum_n a_n$ with nonincreasing positive terms.

It is obvious that not every operator $A \in \mathfrak{S}_\infty$ for which condition (7.12) is fulfilled with a given p (> 0) belongs to $\mathfrak{S}_p$; however for $p > 1$ all such operators form a certain s. n. ideal (cf. §14).

We note without proof the following matrix criterion.

If for the operator $A \in \mathfrak{R}$ *and for some orthonormal basis* $\{\phi_j\}_1^\infty$ *and* $p \leqq 2$ *at least one of the conditions*

$$\sum_{j=1}^{\infty} |A\phi_j|^p < \infty, \qquad \sum_{j,k=1}^{\infty} |(A\phi_j, \phi_k)|^p < \infty$$

is fulfilled, then $A \in \mathfrak{S}_p$, *and*

$$|A|_p^p \leqq \sum_{j=1}^{\infty} |A\phi_j|^p \leqq \sum_{j,k=1}^{\infty} |(A\phi_j, \phi_k)|^p.$$

On the other hand, if $A \in \mathfrak{S}_p$ *for* $p \geqq 2$ *then for any orthonormal basis* $\{\phi_j\}_1^\infty$ *of the space* $\mathfrak{H}$

$$\sum_{j,k=1}^{\infty} |(A\phi_j, \phi_k)|^p \leqq \sum_{j=1}^{\infty} |A\phi_j|^p \leqq |A|_p^p.$$

For the case $p = 1$ the first part of this result was pointed out by W. F. Stinespring [1] (see also Gel'fand and Vilenkin [1]). In the general case it was obtained by Gohberg and Markus [2, 3] (see also Dunford and Schwartz [2], p. 1116).

§8. Nuclear operators

From now on an operator A will be called *nuclear* if it belongs to $\mathfrak{S}_1$, i.e. if

$$\sum_j s_j(A) < \infty.$$

Another characterization of a nuclear operator will be given below, which makes it possible to introduce the notion of a trace for such operators.

1. Let us agree to say that a bounded linear operator A, acting in $\mathfrak{H}$, has a *finite matrix trace* if, for any orthonormal basis $\{\phi_j\}_1^\infty$ of the space $\mathfrak{H}$, the series

$$\sum_{j=1}^{\infty} (A\phi_j, \phi_j) \tag{8.1}$$

converges.

Since any permutation of an orthonormal basis makes it once again an orthonormal basis, for an operator A with a finite matrix trace the series (8.1) converges absolutely for any orthonormal basis $\{\phi_j\}_1^\infty$.

LEMMA 8.1. *Let H be a bounded linear nonnegative operator. Then the sum*

$$\sum_{j=1}^{\infty} (H\chi_j, \chi_j) \tag{8.2}$$

has the same value (finite or infinite) for any orthonormal basis $\{\chi_j\}_1^\infty$ of the space $\mathfrak{H}$. The operator H belongs to $\mathfrak{S}_1$ if and only if this value is finite.

PROOF. If for some orthonormal basis $\{\chi_j\}_1^\infty$ the sum (8.2) is finite, then from the relations

$$\left| H^{1/2}f - \sum_{j=1}^{n} (f, \chi_j) H^{1/2}\chi_j \right| = \left| \sum_{j=n+1}^{\infty} (f, \chi_j) H^{1/2}\chi_j \right|$$
$$\leq \sum_{j=n+1}^{\infty} |H^{1/2}\chi_j| \, |(f, \chi_j)| \leq \left(\sum_{j=n+1}^{\infty} |H^{1/2}\chi_j|^2 \right)^{1/2} |f|$$

and

$$\sum_{j=1}^{\infty} |H^{1/2}\chi_j|^2 = \sum_{j=1}^{\infty} (H\chi_j, \chi_j) < \infty$$

it follows that the sequence of finite-dimensional operators

$$K_n = \sum_{j=1}^{n} (\cdot, \chi_j) H^{1/2}\chi_j \qquad (n = 1, 2, \cdots)$$

converges in the uniform norm to the operator $H^{1/2}$. Thus the operator $H^{1/2}$, and with it the operator H, is completely continuous.

For every other orthonormal basis $\{\phi_j\}_1^\infty$ we will have

$$\sum_{k=1}^{\infty} |H^{1/2}\chi_k|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |(H^{1/2}\chi_k, \phi_j)|^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |(H^{1/2}\phi_j, \chi_k)|^2$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |(H^{1/2}\phi_j, \chi_k)|^2 = \sum_{j=1}^{\infty} |H^{1/2}\phi_j|^2 = \sum_{j=1}^{\infty} (H\phi_j, \phi_j),$$

so that

$$\sum_{j=1}^{\infty} (H\phi_j, \phi_j) = \sum_{j=1}^{\infty} (H\chi_j, \chi_j). \tag{8.3}$$

In particular, choosing for $\{\phi_j\}_1^\infty$ a complete orthonormal system of eigenvectors of the operator H, we obtain from (8.3)

$$\sum_j \lambda_j(H) = \sum_{j=1}^{\infty} (H\chi_j, \chi_j) < \infty,$$

i.e., $H \in \mathfrak{S}_1$. The lemma is proved.

THEOREM 8.1. *In order that the bounded linear operator A have a finite matrix trace, it is necessary and sufficient that it be nuclear. If $A \in \mathfrak{S}_1$, then the sum*

$$\sum_{j=1}^{\infty} (A\chi_j, \chi_j)$$

does not depend upon the choice of the orthonormal basis $\{\chi_j\}_1^\infty$ of the space $\mathfrak{H}$.

This sum is denoted by $\operatorname{sp} A$ and is called the (*matrix*) *trace* of the operator A.

PROOF. We shall first prove the sufficiency of the hypothesis of the theorem. Let $A \in \mathfrak{S}_1$ and let

$$A = \sum_{j=1}^{\infty} s_j(\cdot, \phi_j)\psi_j \qquad (s_j = s_j(A);\ j = 1, 2, \cdots) \tag{8.4}$$

be its expansion in a Schmidt series. We denote by $\{\chi_j\}_1^\infty$ any orthonormal basis of the space $\mathfrak{H}$. Then, by inequality (7.9),

$$\sum_{k=1}^{\infty} |(A\chi_k, \chi_k)| \leqq \sum_{k=1}^{\infty} s_k(A),$$

and so the operator A has a finite matrix trace.

It is worth mentioning that in the case being considered the last inequality can be obtained directly by means of the relations

$$\sum_{k=1}^{\infty} |(A\chi_k, \chi_k)| \leqq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} s_j |(\chi_k, \phi_j)|\,|(\psi_j, \chi_k)|$$

(8.5)
$$\leqq \sum_{j=1}^{\infty} s_j \left(\sum_{k=1}^{\infty} |(\chi_k, \phi_j)|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |(\psi_j, \chi_k)|^2\right)^{1/2} = \sum_{j=1}^{\infty} s_j(A).$$ [7]

According to (8.4)

$$\sum_{k=1}^{\infty} (A\chi_k, \chi_k) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} s_j(\chi_k, \phi_j)(\psi_j, \chi_k)$$
$$= \sum_{j=1}^{\infty} s_j \sum_{k=1}^{\infty} (\chi_k, \phi_j)(\psi_j, \chi_k),$$

and since

$$\sum_{k=1}^{\infty} (\chi_k, \phi_j)(\psi_j, \chi_k) = (\psi_j, \phi_j) \qquad (j = 1, 2, \cdots),$$

it follows that

$$\sum_{k=1}^{\infty} (A\chi_k, \chi_k) = \sum_{j=1}^{\infty} s_j(A)(\psi_j, \phi_j).$$

Thus the left side of this relation does not depend upon the choice of the basis $\{\chi_j\}$.

Let us go on to the proof of the necessity of the hypothesis of the theorem. We first consider the case in which the operator A $(\in \mathfrak{R})$, which has a finite matrix trace, is selfadjoint. Let us denote by $E(\lambda)$ the spectral function of A. As is known, the subspaces $E(0)\mathfrak{H} = \mathfrak{L}_-$ and $(I - E(0))\mathfrak{H} = \mathfrak{L}_+$ are invariant with respect to A, and both of the operators $A_- = -AE(0)$ and $A_+ = A(I - E(0))$ are nonnegative.

Let us denote by $\{\omega_j\}$ and $\{\psi_j\}$ orthonormal bases of the subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ respectively. From the finiteness of the matrix trace of the operator A we get

$$\sum_{j=1}^{\infty} (A_+\omega_j, \omega_j) < \infty, \quad \sum_{j=1}^{\infty} (A_-\psi_j, \psi_j) < \infty;$$

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[7] In particular, if for the χ_k one chooses an orthonormal Schur system, the inequality (8.5) gives

$$\sum_j |(A\omega_j, \omega_j)| = \sum_j |\lambda_j(A)| \leqq |A|_1,$$

which we already know from (7.7); cf. also Remark 7.1.

consequently, by Lemma 8.1, $A_+, A_- \in \mathfrak{S}_1$. Since $A = A_+ - A_-$, we have $A \in \mathfrak{S}_1$.

Let us consider the general case. Let $A \in \mathfrak{R}$ be an operator with finite matrix trace. Obviously the adjoint A^* has finite matrix trace, and consequently so do the Hermitian components

$$A_{\mathscr{R}} = \frac{1}{2}(A + A^*), \quad A_{\mathscr{I}} = \frac{1}{2i}(A - A^*).$$

Thus, by what was proved earlier, $A_{\mathscr{R}}, A_{\mathscr{I}} \in \mathfrak{S}_1$, and so $A = A_{\mathscr{R}} + iA_{\mathscr{I}} \in \mathfrak{S}_1$. The theorem is proved.

2. It is known that for operators in a finite-dimensional space the functional $\operatorname{sp} A$ has the following properties:

1) $$\operatorname{sp}(\alpha A + \beta B) = \alpha \operatorname{sp} A + \beta \operatorname{sp} B;$$
2) $$\operatorname{sp} A^* = \overline{\operatorname{sp} A};$$
3) $$\operatorname{sp}(AB) = \operatorname{sp}(BA);$$
4) $$\operatorname{sp}(S^{-1}AS) = \operatorname{sp} A;$$
5) $$\operatorname{sp} A = \sum_1^{\nu(A)} \lambda_j(A).$$

These properties can be generalized to the infinite-dimensional case. The first two properties, for operators $A, B \in \mathfrak{S}_1$, are obvious.

THEOREM 8.2. *If the operators $A \in \mathfrak{S}_\infty$ and $B \in \mathfrak{R}$ have the properties that $AB \in \mathfrak{S}_1$ and $BA \in \mathfrak{S}_1$, then*

$$\operatorname{sp}(AB) = \operatorname{sp}(BA). \tag{8.6}$$

In particular, if $A \in \mathfrak{S}_p$ $(1 \leq p \leq \infty)$ and $B \in \mathfrak{S}_q$ $(p^{-1} + q^{-1} = 1)$, then (8.6) *holds.*

PROOF. In fact, let

$$A = \sum_{j=1}^\infty s_j(\cdot, \phi_j)\psi_j$$

be the Schmidt expansion of the operator A. Then

$$BA = \sum_{j=1}^\infty s_j(\cdot, \phi_j)B\psi_j$$

and

$$\operatorname{sp}(BA) = \sum_{j=1}^\infty (BA\phi_j, \phi_j) = \sum_{j=1}^\infty s_j(B\psi_j, \phi_j). \tag{8.7}$$

On the other hand,

$$AB = \sum_{j=1}^{\infty} s_j(\cdot, B^*\phi_j)\psi_j$$

and

$$\text{(8.8)} \qquad \mathrm{sp}(AB) = \sum_{j=1}^{\infty} (AB\psi_j, \psi_j) = \sum_{j=1}^{\infty} s_j(B\psi_j, \phi_j).$$

Comparing (8.7) and (8.8), we obtain (8.6).

COROLLARY 8.1. *If $A \in \mathfrak{S}_1$ and if $S \in \mathfrak{R}$ is an invertible operator, then*

$$\mathrm{sp}(S^{-1}AS) = \mathrm{sp}(A).$$

Indeed, by (8.6)

$$\mathrm{sp}(S^{-1}AS) = \mathrm{sp}(ASS^{-1}) = \mathrm{sp}(A).$$

THEOREM 8.3. *If A and B are selfadjoint operators, one of which is completely continuous, and if $AB \in \mathfrak{S}_1$, then* $\mathrm{sp}(AB)$ *is a real number. If moreover the operators A and B are nonnegative, then*

$$\text{(8.9)} \qquad \mathrm{sp}(AB) \geqq 0,$$

and $\mathrm{sp}(AB)$ *equals zero if and only if the operators A and B are orthogonal, i.e.*

$$AB = BA = 0.$$

PROOF. Let B be a completely continuous operator and let

$$B = \sum_{j=1}^{r(B)} \lambda_j(\cdot, e_j)e_j \qquad (\lambda_j = \lambda_j(B))$$

be its spectral decomposition. Then

$$AB = \sum_{j=1}^{r(B)} \lambda_j(\cdot, e_j)Ae_j$$

and

$$\text{(8.10)} \qquad \mathrm{sp}(AB) = \sum_{j=1}^{r(B)} (ABe_j, e_j) = \sum_{j=1}^{r(B)} \lambda_j(Ae_j, e_j).$$

Consequently $\mathrm{sp}(AB)$ is a real number. If the operators A and B are nonnegative, then

$$\text{(8.11)} \qquad \lambda_j > 0 \quad \text{and} \quad (Ae_j, e_j) \geqq 0 \qquad (j = 1, 2, \cdots, r(B)),$$

and so the inequality (8.9) is fulfilled.

If $\mathrm{sp}(AB) = 0$, then from (8.10) and (8.11) it follows that $(Ae_j, e_j) = 0$. By the nonnegativeness of the operator A, it follows that $Ae_j = 0$

$(j = 1, 2, \cdots, r(B))$. This means that $AB = 0$ and $BA = (AB)^* = 0$. The theorem is proved.

REMARK 8.1. We call attention to the fact that the definition itself of the nonnegativeness of an operator $A \in \mathfrak{R}$ can be written in the form $\operatorname{sp}(AK) \geqq 0$, where K is an arbitrary nonnegative one-dimensional operator. In fact, if $K = (\cdot, \phi)\phi$, then $\operatorname{sp}(AK) = (A\phi, \phi)$.

3. Although the operators of the class $\mathfrak{S}_1$ attracted the attention of mathematicians rather long ago, the following of their fundamental properties was only discovered in 1958.

THEOREM 8.4 (V. B. LIDSKIĬ [6]). *If $A \in \mathfrak{S}_1$, then the matrix trace of the operator A coincides with its spectral trace*:

$$\operatorname{sp} A = \sum_{j=1}^{\nu(A)} \lambda_j(A).$$

PROOF.[8] We shall first prove that if A is a Volterra operator, then $\operatorname{sp} A = 0$. Let P_n $(n = 1, 2, \cdots)$ be an increasing sequence of finite-dimensional orthoprojectors which converges strongly to the unit operator, let $\dim \mathfrak{R}(P_n) = n$ and let $\lambda_j^{(n)} = \lambda_j(P_n A P_n)$ $(j = 1, 2, \cdots, n;$ $n = 1, 2, \cdots)$. We introduce the function $D_n(\lambda)$ $(n = 1, 2, \cdots)$, setting

$$D_n(\lambda) = \prod_{j=1}^{n} (1 - \lambda\lambda_j^{(n)}).$$

The logarithmic derivative of the function $D_n(\lambda)$ has the form

$$\frac{D_n'(\lambda)}{D_n(\lambda)} = -\sum_{j=1}^{n} \frac{\lambda_j^{(n)}}{1 - \lambda\lambda_j^{(n)}};$$

consequently, for $|\lambda\lambda_1^{(n)}| < 1$

$$\frac{D_n'(\lambda)}{D_n(\lambda)} = -\sum_{j=1}^{\infty} S_j^{(n)} \lambda^{j-1},$$

where

$$S_j^{(n)} = \mathbf{sp}\,[(P_n A P_n)^j] = \sum_{k=1}^{n} (\lambda_k^{(n)})^j \qquad (j = 1, 2, \cdots).$$

From the relations

$$\left| \sum_{j=1}^{n} (\lambda_j^{(n)})^k \right| \leqq \sum_{j=1}^{n} |\lambda_j^{(n)}|^k \leqq |\lambda_1^{(n)}|^{k-1} \sum_{j=1}^{n} |\lambda_j^{(n)}| \leqq |A|_1 |\lambda_1^{(n)}|^{k-1}$$

[8] The proof presented here is simpler than the original proof; it was worked out together with the author of the theorem. For clarity the trace of a finite matrix is denoted below by **sp**.

it follows that

$$\left|\frac{D_n'(\lambda)}{D_n(\lambda)} + \operatorname{sp} A\right| \leqq |S_1^{(n)} - \operatorname{sp} A| + \frac{\epsilon_n|\lambda|\,|A|_1}{1 - \epsilon_n|\lambda|}, \tag{8.12}$$

where $\epsilon_n = |\lambda_1^{(n)}|$.

Since A is a Volterra operator and the operators P_nAP_n tend to A in the uniform norm, according to Theorem I.4.2

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} |\lambda_1^{(n)}| = 0.$$

Moreover, by the definition of the trace,

$$\lim_{n\to\infty} S_1^{(n)} = \operatorname{sp} A.$$

Thus, by virtue of (8.12) the sequence of functions $D_n'(\lambda)/D_n(\lambda)$ $(n = 1, 2, \cdots)$ converges to $a = -\operatorname{sp} A$ uniformly in any bounded region:

$$\lim_{n\to\infty} \frac{D_n'(\lambda)}{D_n(\lambda)} = -\operatorname{sp} A = a. \tag{8.13}$$

Integrating this, we obtain

$$\lim_{n\to\infty} D_n(\lambda) = e^{a\lambda}. \tag{8.14}$$

Let us now estimate the function $D_n(\lambda)$ and its limit as $n \to \infty$ from other considerations. Obviously

$$|D_n(\lambda)| \leqq \prod_{j=1}^{n} (1 + |\lambda_j^{(n)}|\,|\lambda|).$$

By the inequalities of Corollary II.3.1

$$\prod_{j=1}^{n} (1 + |\lambda_j^{(n)}|\,|\lambda|) \leqq \prod_{j=1}^{n} (1 + s_j^{(n)}|\lambda|) \qquad (n = 1, 2, \cdots),$$

where $s_j^{(n)} = s_j(P_nAP_n)$, and so

$$D_n(\lambda) \leqq \prod_{j=1}^{n} (1 + s_j^{(n)}|\lambda|).$$

Taking into account that $s_j^{(n)} \leqq s_j = s_j(A)$, we obtain

$$|D_n(\lambda)| \leqq \prod_{j=1}^{\infty} (1 + s_j|\lambda|) \leqq \prod_{j=1}^{N} (1 + s_j|\lambda|) \exp\left[|\lambda| \sum_{j=N+1}^{\infty} s_j\right] \tag{8.15}$$
$$(n = 1, 2, \cdots).$$

Let us assume that $a = -\operatorname{sp} A \neq 0$; then, setting

$$\lambda = \rho e^{-i \arg a} \qquad (0 < \rho < \infty)$$

and choosing N so that

$$\sum_{j=N+1}^{\infty} s_j < \frac{|a|}{2},$$

we obtain from (8.14) and (8.15) the inequality

$$\exp\left[\rho \frac{|a|}{2}\right] \leqq \prod_{j=1}^{N} (1 + s_j \rho) \qquad (0 < \rho < \infty),$$

which is not possible. Thus,

$$\operatorname{sp} A = 0.$$

So for a Volterra operator A the theorem is proved.

Let us consider the general case. Let A be any operator from $\mathfrak{S}_1$. We denote by $\mathfrak{E}_A$ the closed linear hull of all root vectors of the operator A corresponding to nonzero eigenvalues. Let $\{\omega_j\}_1^{\nu(A)}$ be an orthonormal Schur system for the operator A (cf. Lemma I.4.1). Then

$$(A\omega_j, \omega_j) = \lambda_j(A) \qquad (j = 1, 2, \cdots, \nu(A)).$$

Denoting by $\hat{A}$ the operator induced by A in $\mathfrak{E}_A$, we obtain

$$\operatorname{sp} \hat{A} = \sum_{j=1}^{\nu(A)} (A\omega_j, \omega_j) = \sum_{j=1}^{\nu(A)} \lambda_j(A).$$

Since the subspace $\mathfrak{E}_A$ is invariant with respect to A, we have

$$A = PAP + PAQ + QAQ,$$

where P is the orthogonal projector onto the subspace $\mathfrak{E}_A$, and $Q = I - P$. By Lemma I.4.2, QAQ is a Volterra operator, and so, as was proved,

$$\operatorname{sp}(QAQ) = 0.$$

Moreover,

$$\operatorname{sp}(PAQ) = \operatorname{sp}(AQP) = 0 \quad \text{and} \quad \operatorname{sp}(PAP) = \operatorname{sp}(\hat{A}).$$

Consequently

$$\operatorname{sp} A = \operatorname{sp}(PAP) + \operatorname{sp}(PAQ) + \operatorname{sp}(QAQ) = \operatorname{sp} \hat{A},$$

i.e.

$$\operatorname{sp} A = \sum_{j=1}^{\nu(A)} \lambda_j(A).$$

The theorem is proved.

4. Obviously $\operatorname{sp} A$ is a linear functional on the Banach space $\mathfrak{S}_1$. The following result shows that this functional is continuous and has norm equal to unity.

THEOREM 8.5. *If* $A \in \mathfrak{S}_1$ $(A \neq 0)$, *then*

$$|\operatorname{sp} A| \leqq |A|_1, \tag{8.16}$$

and equality holds if and only if, for $\theta = \arg \operatorname{sp} A$, *the operator* $e^{-i\theta}A$ *is nonnegative.*

PROOF. In fact, by virtue of Theorem 8.4 and the bound (7.7)

$$|\operatorname{sp} A| \leqq \sum_{j=1}^{\nu(A)} |\lambda_j(A)| \leqq \sum_{j=1}^{\infty} s_j(A) = |A|_1.$$

If equality holds in (8.16), then

$$\begin{aligned} \left| \sum_{j=1}^{\nu(A)} \lambda_j(A) \right| &= \sum_{j=1}^{\nu(A)} |\lambda_j(A)|, \\ \sum_{j=1}^{\nu(A)} |\lambda_j(A)| &= \sum_{j=1}^{\infty} s_j(A). \end{aligned} \tag{8.17}$$

According to Theorem II.3.1, it follows from the second of the equalities (8.17) that A is a normal operator. On the other hand, the first equality shows that all the nonzero eigenvalues of A have the same argument θ. Consequently the operator $e^{-i\theta}A$ is nonnegative. It is obvious, conversely, that if the operator $e^{-i\theta}A$ is nonnegative, then equality holds in (8.16). The theorem is proved.

It admits the following sharpening.

THEOREM 8.6. *Let* $A \in \mathfrak{S}_1$ *and let* $\{\phi_j\}_1^{\omega}$ $(\omega \leqq \infty)$ *be some (generally speaking, incomplete) orthonormal system of vectors. Then*

$$\sum_{j=1}^{\omega} |(A\phi_j, \phi_j)| \leqq |A|_1, \tag{8.18}$$

and equality holds if and only if the operator A has the following properties:

a) *A is identically zero on the subspace* $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{L}$, *where* $\mathfrak{L}$ *is the closed linear hull of those* ϕ_j $(j = 1, 2, \cdots, \omega)$ *for which* $(A\phi_j, \phi_j) \neq 0$;

b) *the operator UA is nonnegative, where U is a unitary operator such that*

$$U\phi_j = e^{-i\theta_j}\phi_j, \quad \theta_j = \arg(A\phi_j, \phi_j) \qquad (\phi_j \in \mathfrak{L}). \tag{8.19}$$

The equality

$$\left| \sum_{j=1}^{\omega} (A\phi_j, \phi_j) \right| = |A|_1 \tag{8.20}$$

holds if and only if the operator A has the property a) *and the operator $e^{i\theta}A$ is nonnegative for $\theta = \arg \operatorname{sp} A$.*

PROOF. In fact, the relation (8.18) follows from §4.3, Chapter II. Let us consider the case in which equality holds in (8.18). One can then assert that

$$(A\phi, \phi) = 0 \qquad (\phi \in \mathfrak{N}).$$

Now let U be a unitary operator having the property (8.19). Then

$$(UA\phi_j, \phi_j) = |(A\phi_j, \phi_j)| \qquad (\phi_j \in \mathfrak{L})$$

and consequently

$$\operatorname{sp}(UA) = \sum_{j=1}^{\omega} |(A\phi_j, \phi_j)| = \sum_{\phi_j \in \mathfrak{L}} |(A\phi_j, \phi_j)| = |A|_1.$$

Since $|UA|_1 = |A|_1$, we obtain the equality

$$\operatorname{sp}(UA) = |UA|_1.$$

From Theorem 8.5 we conclude that UA is nonnegative and $\mathfrak{L} = \overline{\mathfrak{R}(UA)}$. Hence

$$UA\phi = 0, \quad A\phi = 0 \qquad (\phi \in \mathfrak{N}).$$

Thus the operator A has the properties a) and b). It is easily seen that, conversely, if A has the properties a) and b), then equality in (8.18) holds for it.

We now consider the case in which (8.20) holds. Then equality will hold in (8.18), and

$$\left| \sum_{j=1}^{\omega} (A\phi_j, \phi_j) \right| = \sum_{j=1}^{\omega} |(A\phi_j, \phi_j)|.$$

It follows that all of the numbers θ_j are equal. Consequently one can put $U = e^{-i\theta}I$. The theorem is proved.

Theorem 8.6, in turn, admits the following generalization.

THEOREM 8.7. *Let $A \in \mathfrak{S}_1$ and let $\{P_j\}_1^{\omega}$ $(\omega \leq \infty)$ be a system of mutually orthogonal orthoprojectors. Then*

$$\left| \sum_{k=1}^{\omega} P_k A P_k \right|_1 = \sum_{k=1}^{\omega} |P_k A P_k|_1 \leq |A|_1, \tag{8.21}$$

and equality holds if and only if the following conditions are fulfilled:

a) $AQ = 0$, *where* $Q = I - \sum_{j=1}^{\omega} P_j$;

b) *there exists a partially isometric operator* U *which commutes with all the projectors* P_k $(k = 1, 2, \cdots, \omega)$ *and is such that*

$$UA = (A^*A)^{1/2}, \quad UP_kAP_k = (P_kA^*P_kAP_k)^{1/2} \quad (k = 1, 2, \cdots, \omega). \tag{8.22}$$

PROOF. The relation (8.21) follows directly from the general Theorem II.5.2. Consequently it remains to verify the conditions under which equality is achieved in (8.21).

Suppose that the conditions a) and b) are fulfilled; then

$$\sum_{k=1}^{\omega} |P_kAP_k|_1 = \sum_{k=1}^{\omega} |UP_kAP_k|_1 = \sum_{k=1}^{\omega} \operatorname{sp}(P_kUAP_k) = \operatorname{sp} UA = |A|_1,$$

i.e. equality holds in (8.21).

Conversely, suppose that equality holds in (8.21). Then it follows from Remark II.5.2 that condition a) is fulfilled. Let $P_kAP_k = U_k^*H_k$ be the polar representation of the operator P_kAP_k in the subspace $\Re(P_k)$.

The partially isometric operator U, defined by

$$U^* = \sum_{k=1}^{\omega} U_k^*P_k + Q,$$

has the property

$$\operatorname{sp}(UA) = \sum_{k=1}^{\omega} \operatorname{sp}(U_kP_kAP_k) = \sum_{k=1}^{\omega} |P_kAP_k|_1 = |A|_1. \tag{8.23}$$

Since

$$\operatorname{sp}(UA) \leqq |UA|_1 \leqq |A|_1,$$

it follows from (8.23) that

$$\operatorname{sp}(UA) = |UA|_1 \quad \text{and} \quad |UA|_1 = |A|_1.$$

From the first equality it follows by Theorem 8.5 that the operator UA is nonnegative; from the second equality and the relation

$$(UA)^2 = A^*U^*UA \leq A^*A$$

it follows that $UA = (A^*A)^{1/2}$. The theorem is proved.

§9. Hilbert-Schmidt operators

1. The operators of the class $\mathfrak{S}_2$ are also called *Hilbert-Schmidt operators*. According to Lemma 8.1, for any nonnegative operator H the sum

$$\sum_{j=1}^{\infty} (H\phi_j, \phi_j) \tag{9.1}$$

assumes the same value (finite or infinite) for any orthonormal basis $\{\phi_j\}_1^\infty$. For the case in which the series (9.1) converges, its sum was denoted by $\operatorname{sp} H$. We maintain this notation also for the case in which the sum of the series (9.1) is infinite.

Under this condition, we can state that

A bounded linear operator A is a Hilbert-Schmidt operator if and only if

$$\operatorname{sp}(A^*A) < \infty. \tag{9.2}$$

The necessity of this condition is clear from the definition of the class $\mathfrak{S}_2$, and the sufficiency, from the fact that if A is a bounded operator and (9.2) holds, then according to Lemma 8.1 $A^*A \in \mathfrak{S}_1$, and then

$$\operatorname{sp}(A^*A) = \sum_{j=1}^{\nu(A)} \lambda_j(A^*A) = \sum_{j=1}^{\infty} s_j^2(A) < \infty.$$

To verify the condition (9.2) it is useful to keep in mind that for any bounded operator A and any two orthonormal bases $\{\phi_j\}_1^\infty$, $\{\psi_j\}_1^\infty$

$$\operatorname{sp}(A^*A) = \sum_{j=1}^{\infty} |A\phi_j|^2 = \sum_{j,k=1}^{\infty} |(A\phi_j, \phi_k)|^2 = \sum_{j,k=1}^{\infty} |(A\psi_j, \psi_k)|^2.$$

THEOREM 9.1 (I. SCHUR [1]). *If $A \in \mathfrak{S}_2$, then*

$$\sum_{j=1}^{\nu(A)} |\lambda_j(A)|^2 \leqq \operatorname{sp}(A^*A). \tag{9.3}$$

The equal sign holds if and only if A is a normal operator.

This theorem is a consequence of Theorem II.3.1 of H. Weyl.

However, Schur [1] established his theorem long before the appearance of Weyl's result (in 1909), and moreover by means of a simple argument which used, for the first time, the method of reducing a matrix (operator) to triangular form with the help of a unitary transformation. We shall present Schur's elementary proof.

PROOF. According to Schur's lemma (Lemma I.4.1) one can choose, in the closed linear hull $\mathfrak{E}_A$ of all the root lineals $\mathfrak{L}_{\lambda_j}(A)$ $(j = 1, 2, \cdots, \nu(A))$, an orthonormal basis ω_j $(j = 1, 2, \cdots, \nu(A))$ such that

$$A\omega_j = a_{j1}\omega_1 + a_{j2}\omega_2 + \cdots + \lambda_j(A)\omega_j.$$

Since

(9.4)
$$|A\omega_j|^2 = \sum_{k=1}^{j-1} |a_{jk}|^2 + |\lambda_j|^2,$$

it follows that

(9.5)
$$\sum_{j=1}^{\nu(A)} |\lambda_j|^2 \leqq \sum_{j=1}^{\nu(A)} |A\omega_j|^2 \leqq \mathrm{sp}(A^*A).$$

From (9.4) and (9.5) it follows that the equal sign in (9.3) will hold if and only if

$$a_{jk} = 0 \ \ (k = 1, 2, \cdots, j-1;\ j = 1, 2, \cdots, \nu(A)) \ \text{ and } \ A\phi = 0 \quad (\phi \in \mathfrak{E}_A^{\perp}),$$

and this is equivalent to the normality of the operator A.

2. It can be verified without difficulty that the formula

$$(A, B) = \mathrm{sp}(AB^*)$$

defines a scalar product in $\mathfrak{S}_2$.

Since

$$(A, A)^{1/2} = (\mathrm{sp}(AA^*))^{1/2} = |A|_2 \qquad (A \in \mathfrak{S}_2),$$

the scalar product which has been introduced turns $\mathfrak{S}_2$ into a complete Hilbert space.

We note the identity

$$(|A_{\mathscr{R}}|_2)^2 + (|A_{\mathscr{J}}|_2)^2 = (|A|_2)^2 \qquad (A \in \mathfrak{S}_2)$$

or, what is the same,

$$\mathrm{sp}\, A_{\mathscr{R}}^2 + \mathrm{sp}\, A_{\mathscr{J}}^2 = \mathrm{sp}\, A^*A.$$

The last equality is an immediate consequence of the identity $A_{\mathscr{R}}^2 + A_{\mathscr{J}}^2 = (AA^* + A^*A)/2$.

3. We shall give an analytic description of the Hilbert-Schmidt operators for the case in which the space $\mathfrak{H}$ is realized in the form of a space of vector-functions $L_2^{(r)}(Q)$.

Let Q be some bounded closed set of points in n-dimensional Euclidean space E_n, having positive Lebesgue measure.

Given an r $(= 1, 2, \cdots, \infty)$, one can form a Hilbert space $L_2^{(r)}(Q)$, consisting of all r-dimensional vector-functions $f(x) = \{f_j(x)\}_1^r$, $x \in Q$, with measurable coordinates $f_j(x)$ $(j = 1, 2, \cdots, r)$ such that

$$|f|^2 = \int_Q \sum_{j=1}^{r} |f_j(x)|^2 dx < \infty.$$

The scalar product in $L_2^{(r)}(Q)$ is defined in a natural way by the formula[9]

$$(f,g) = \int_Q g^*(x)\, f(x)\, dx = \int_Q \sum_{j=1}^{r} \overline{g_j(x)}\, f_j(x)\, dx \quad (f,g \in L_2^{(r)}(Q)).$$

We shall form, on the other hand, the Hilbert space $L_2^{(r\times r)}(Q\times Q)$, consisting of matrix-functions $\mathscr{A}(x,y) = \|a_{\mu\nu}(x,y)\|_1^r$ with elements measurable on $Q\times Q$, such that

$$(9.6) \qquad |\mathscr{A}(x,y)|^2 = \int_Q \int_Q \sum_{j,k=1}^{r} |a_{jk}(x,y)|^2 dx\, dy < \infty,$$

in which the scalar product of two elements $\mathscr{A}(x,y)$ and $\mathscr{B}(x,y)$ is defined by the formula

$$(\mathscr{A},\mathscr{B}) = \int_Q \int_Q \sum_{j,k=1}^{r} a_{jk}(x,y)\, \overline{b_{jk}(x,y)}\, dx\, dy.$$

One has the following result.

1. *To every element $\mathscr{A}(x,y) \in L_2^{(r\times r)}(Q\times Q)$ there corresponds a Hilbert-Schmidt operator A, acting in $L_2^{(r)}(Q)$ according to the formula*[10]

$$(9.7) \qquad (Af)(x) = \int_Q \mathscr{A}(x,y)\, f(y)\, dy \qquad (f \in L_2^{(r)}(Q)).$$

The mapping $\mathscr{A}(x,y) \to A$ is an isometric mapping of all of $L_2^{(r\times r)}(Q\times Q)$ onto all of the Hilbert space $\mathfrak{S}_2$ of Hilbert-Schmidt operators, acting in $L_2^{(r)}(Q)$. Thus

$$(9.8) \qquad (\mathscr{A},\mathscr{B}) = \operatorname{sp}(B^*A).$$

PROOF. Since for finite r the proof is rather simple, we shall at once consider the more difficult case $r = \infty$.

For brevity we denote the spaces $L_2^{(r)}(Q)$ and $L_2^{(r\times r)}(Q\times Q)$ by L_2 and $L_2\times L_2$.

By a well-known theorem of Fubini it follows from the condition (9.6) for $r = \infty$ that for almost all $x \in Q$

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[9] Let us clarify that an r-dimensional vector $\xi = \{\xi_j\}_1^r$ is regarded as a column-vector, and then $\xi^* = \{\bar\xi_1, \bar\xi_2, \cdots, \bar\xi_r\}$ is regarded as a row-vector. In this notation the product $\eta^*\xi$ of the vectors ξ and $\eta = \{\eta_j\}_1^r$ is the scalar $\sum_1^r \xi_j\bar\eta_j$, and the product $\xi\eta^*$ is the square $(r\times r)$ matrix $\|\xi_j\bar\eta_k\|_1^r$.

[10] The precise meaning of this formula will be clarified in the course of proving the theorem.

$$\int_Q \sum_{j,k=1}^{\infty} |a_{jk}(x,y)|^2 dy < \infty. \tag{9.9}$$

For those $x \in Q$ for which the series (9.9) converges, the bound

$$\begin{aligned}
&\left(\int_Q \sum_{k=1}^{\infty} |a_{jk}(x,y)|\, |f_k(y)|\, dy \right)^2 \\
&\leqq \left(\int_Q \left(\sum_{k=1}^{\infty} |a_{jk}(x,y)|^2 \sum_{k=1}^{\infty} |f_k(y)|^2 \right)^{1/2} dy \right)^2 \qquad (9.10)\\
&\leqq \int_Q \sum_{k=1}^{\infty} |a_{jk}(x,y)|^2 dy \int_Q \sum_{k=1}^{\infty} |f_k(y)|^2 dy \qquad (j = 1, 2, \cdots)
\end{aligned}$$

is valid for any $f = \{f_j(y)\}_1^\infty \in L_2$. Hence for these x the integrals

$$g_j(x) = \int_Q \sum_{k=1}^{\infty} a_{jk}(x,y)\, f_k(y)\, dy \qquad (j = 1, 2, \cdots) \tag{9.11}$$

are meaningful, and moreover, according to (9.10),

$$\int_Q \sum_{j=1}^{\infty} |g_j(x)|^2 dx \leqq \int_{Q\times Q} \sum_{j,k=1}^{\infty} |a_{jk}(x,y)|^2 dx\, dy \int_Q \sum_{k=1}^{\infty} |f_k(y)|^2 dy < \infty.$$

The last inequality shows that $g = \{g_j\}_1^\infty \in L_2$; the equalities (9.11) define a linear operator $g = Af$, acting in L_2, and

$$|Af| \leqq \sqrt{(\mathscr{A}, \mathscr{A})}\, |f|.$$

The system of equations (9.11) can be written in concise form as

$$g(x) = \int_Q \mathscr{A}(x,y)\, f(y)\, dy. \tag{9.12}$$

We shall show that A is a Hilbert-Schmidt operator and that moreover

$$\mathrm{sp}(A^*A) = (\mathscr{A}, \mathscr{A}). \tag{9.13}$$

Let $\{\phi_j\}_1^\infty$ be some orthonormal basis of the space L_2. We form the matrix-function

$$\mathscr{F}_{jk}(x,y) = \phi_j(y)\phi_k^*(x) = \| \phi_{j\mu}(y)\overline{\phi_{k\nu}(x)} \|_{\mu,\nu=1}^{\infty} \qquad (j, k = 1, 2, \cdots)$$

on $Q \times Q$. It is easily seen that $\mathscr{F}_{jk}(x,y) \in L_2 \times L_2$ $(j, k = 1, 2, \cdots)$ and that they form an orthonormal system:

$$(\mathscr{F}_{jk}, \mathscr{F}_{j_1 k_1}) = \delta_{jj_1}\delta_{kk_1} \qquad (j, k;\ j_1, k_1 = 1, 2, \cdots).$$

The reader can easily show that the system $\{\mathscr{F}_{jk}(x,y)\}_{j,k=1}^\infty$ is complete in $L_2 \times L_2$.

As soon as this is shown, we have

$$(9.14)\qquad (\mathscr{A}, \mathscr{A}) = \sum_{j,k=1}^{\infty} |(\mathscr{A}, \mathscr{F}_{jk})|^2.$$

On the other hand, it is easily seen that

$$(9.15)\qquad (\mathscr{A}, \mathscr{F}_{jk}) = \int_{Q\times Q} \phi_k^*(x)\, \mathscr{A}(x,y)\, \phi_j(y)\, dx\, dy = (A\phi_j, \phi_k) \qquad (k,j = 1,2,\cdots).$$

Thus the equality (9.13) is a consequence of (9.14) and (9.15). Since the mapping $\mathscr{A} \to A$ is linear, it is not hard to conclude from (9.13) that if $\mathscr{A} \to A$ and $\mathscr{B} \to B$, then (9.8) holds.

To complete the proof of the theorem, it remains for us to show that any Hilbert-Schmidt operator in L_2 can be represented in the form (9.11) with some kernel $\mathscr{A}(x,y) \in L_2 \times L_2$. To do this we shall again use some orthonormal basis $\{\phi_j\}_1^\infty$ of the space L_2 and the corresponding orthonormal basis $\{\mathscr{F}_{jk}\}$ of the space $L_2 \times L_2$.

If A is a Hilbert-Schmidt operator, then

$$\mathrm{sp}(A^*A) = \sum_{j,k=1}^{\infty} |\gamma_{jk}|^2, \ \gamma_{jk} = (A\phi_j, \phi_k) \qquad (j,k = 1,2,\cdots).$$

We form the kernel $\mathscr{A}_0(x,y) \in L_2 \times L_2$, setting

$$(9.16)\qquad \mathscr{A}_0(x,y) = \sum_{j,k=1}^{\infty} \gamma_{jk} \mathscr{F}_{jk}(x,y) = \sum_{j,k=1}^{\infty} \gamma_{jk} \phi_j(x)\, \phi_k^*(y),$$

where the infinite series on the right side converges in the sense of $L_2 \times L_2$.

We denote by A_0 the Hilbert-Schmidt operator, generated in L_2 by the kernel $\mathscr{A}_0(x,y)$ according to the formula (9.7). According to (9.16)

$$(\mathscr{A}_0, \mathscr{F}_{jk}) = \gamma_{jk} = (A\phi_j, \phi_k) \qquad (j,k = 1,2,\cdots),$$

and according to (9.7):

$$(\mathscr{A}_0, \mathscr{F}_{jk}) = (A_0\phi_j, \phi_k) \qquad (j,k = 1,2,\cdots),$$

so that $(A\phi_j, \phi_k) = (A_0\phi_j, \phi_k)$ $(j,k = 1,2,\cdots)$, whence $A = A_0$. The result is proved.

REMARK 9.1. If A is a selfadjoint operator acting in L_2, then the series (9.16), which gives its kernel $\mathscr{A}(x,y)$, assumes a particularly simple form, if for the orthonormal system $\{\phi_j\}$ one chooses a complete orthonormal system of eigenvectors of the operator A;[11)] in this case

[11)] With regard also for a zero eigenvalue of the operator A.

$$A\phi_j = \lambda_j(A)\phi_j, \quad (\phi_j, \phi_k) = \delta_{jk} \qquad (j,k = 1,2,\cdots).$$

Then

$$\gamma_{jk} = (A\phi_j, \phi_k) = \lambda_j(A)\delta_{jk} \qquad (j,k = 1,2,\cdots)$$

and formula (9.16) gives

$$\mathscr{A}_0(x,y) = \sum_{j=1}^{\infty} \lambda_j(A)\phi_j(x)\phi_j^*(y),$$

where the series on the right converges in the norm of the space $L_2 \times L_2$.

§10. Tests for the nuclearity of integral operators and formulas for calculating the trace

In view of the special role of nuclear operators, we shall present in this section tests for the nuclearity of integral operators and shall indicate formulas for calculating their trace.

1. Let $\mathscr{I}$ be a closed, open or half-open interval of the real line, having finite or infinite length. A continuous kernel $\mathscr{A}(t,s)\,(t,s \in \mathscr{I})$ is said to be *Hermitian nonnegative* on $\mathscr{I}$ if for any t_j $(j = 1,2,\cdots,n;$ $n = 1,2,\cdots)$ the quadratic form

$$\sum_{j,k=1}^{n} \mathscr{A}(t_j,t_k)\xi_k\bar{\xi}_j$$

is nonnegative.

This condition automatically includes within itself the requirement that the kernel $\mathscr{A}(t,s)$ be Hermitian: $\mathscr{A}(t,s) = \overline{\mathscr{A}(s,t)}$.

It is easily shown that a continuous kernel $\mathscr{A}(t,s)$ is Hermitian nonnegative if and only if

$$\int_{\mathscr{I}}\int_{\mathscr{I}} \mathscr{A}(t,s)\phi(s)\overline{\phi(t)}\,ds\,dt \geqq 0$$

for any continuous function $\phi(t)\,(t \in \mathscr{I})$ which vanishes in a neighborhood of the endpoints of $\mathscr{I}$ (this requirement is necessary for the convergence of the double integral).

Let us first consider the case in which $\mathscr{I} = [a,b]$ is a finite closed interval.

We consider in $C(a,b)$ the integral equation

$$\phi(t) = \lambda\int_a^b \mathscr{A}(t,s)\phi(s)\,d\sigma(s) \tag{10.1}$$

with a continuous Hermitian nonnegative kernel $\mathscr{A}(t,s)$ on $[a,b]$; here $\sigma(s)$ $(a \leqq s \leqq b)$ is some nondecreasing function with $\operatorname{Var}\sigma = \sigma(b)$

$\sigma(a) > 0$. The ordinary Hilbert-Schmidt-Mercer theory can be extended to such equations.

The equation (10.1) always gives rise to a system of positive eigenvalues $\{\lambda_j\}$ and a corresponding system $\{\phi_j(t)\}$ of continuous eigenfunctions, orthonormal with respect to the measure σ:

$$\int_a^b \phi_j(t)\overline{\phi_k(t)}\,d\sigma(t) = \delta_{jk} \qquad (j,k = 1,2,\cdots). \tag{10.2}$$

Moreover, one has the uniformly convergent expansion (see for example M. G. Kreĭn [2])

$$\mathscr{A}(t,s) = \mathscr{R}(t,s) + \sum_j \frac{\phi_j(t)\overline{\phi_j(s)}}{\lambda_j} \qquad (a \leqq t,s \leqq b), \tag{10.3}$$

where $\mathscr{R}(t,s)\,(a \leqq t,s \leqq b)$[12] is a continuous Hermitian nonnegative kernel, which equals zero if at least one of the points t, s belongs to the set $\mathscr{E}_\sigma$ of points of increase of the function σ.

From (10.2) and (10.3), in particular, follows

$$\sum_j \frac{1}{\lambda_j} = \int_a^b \mathscr{A}(s,s)\,d\sigma(s) < \infty.$$

We form the Hilbert space $L_2^{(\sigma)} = L_2^{(\sigma)}(a,b)$, consisting of all σ-measurable functions $f(t)$ $(a \leqq t \leqq b)$ having σ-integral squares, with scalar product

$$(f,g)_\sigma = \int_a^b f(t)\overline{g(t)}\,d\sigma(t).$$

The kernel $\mathscr{A}(t,s)$ being considered gives rise in $L_2^{(\sigma)}$ to a nonnegative completely continuous operator

$$(Af)(t) = \int_a^b \mathscr{A}(t,s)f(s)\,d\sigma(s). \tag{10.4}$$

By virtue of (10.3) we will have

$$(Af)(t) = \sum_j \frac{c_j}{\lambda_j}\phi_j(t),$$

where $c_j = (f,\phi_j)_\sigma$ $(j = 1,2,\cdots)$. Thus the system $\{\phi_j\}_1^\infty$ will be a complete orthonormal system of eigenvectors of the operator A in $\mathfrak{R}(A)$,

[12] As M. D. Dol'berg [1] first showed (cf. also M. G. Kreĭn [13]), the kernel $\mathscr{R}(t,s)$ is completely determined by the set $\mathscr{E}_\sigma$ and does not depend upon the values of σ on $\mathscr{E}_\sigma$.

and $\{\lambda_j\}$ will be the complete system of characteristic numbers of A. We have arrived at the following conclusion.

1. *If the kernel* $\mathscr{A}(t,s)$ $(a \leqq t, s \leqq b)$ *is continuous and Hermitian nonnegative, then it generates a nonnegative nuclear operator* A *in* $L_2^{(\sigma)}$ *by the formula* (10.4), *and*

$$\operatorname{sp} A = \int_a^b \mathscr{A}(s,s)\, d\sigma(s). \tag{10.5}$$

We shall leave it to the reader to extend this result to all possible types of intervals $\mathscr{I}$ and to the case of σ with infinite variation, i.e., to deduce from 1 the result

2. *Let* $\mathscr{I}$ *be an open or half-open interval (of the real line) of finite or infinite length,* $\sigma(t)$ $(t \in \mathscr{I})$ *a nondecreasing function with* $\operatorname{Var} \sigma \leqq \infty$, *and* $\mathscr{A}(t,s)$ $(t, s \in \mathscr{I})$ *a continuous Hermitian nonnegative kernel. Then the nonnegative operator* A, *defined in* $L_2^{(\sigma)}(\mathscr{I})$ *by the kernel* $\mathscr{A}(t,s)$,

$$(Af)(t) = \int_{\mathscr{I}} \mathscr{A}(t,s) f(s)\, d\sigma(s),$$

will be nuclear if and only if

$$\int_{\mathscr{I}} \mathscr{A}(s,s)\, d\sigma(s) < \infty. \tag{10.5$'$}$$

When this condition is fulfilled, the integral in (10.5′) *yields the trace* $\operatorname{sp} A$.

The concepts and results presented above can be generalized to the case where $\mathscr{I}$ is any compact or locally compact set, and $\sigma(\Delta)$ is some nonnegative measure defined on $\mathscr{I}$. However, everywhere henceforth we shall take $\mathscr{I}$ to be an interval of the real line and $d\sigma(t) = dt$, i.e. we shall consider operators in some space $L_2 = L_2(a,b)$, and the class of operators will be extended by extending the class of kernels $\mathscr{A}(t,s)$.

2. For future considerations we shall need Steklov's smoothing operator S_h $(h > 0)$, defined by

$$(S_h f)(t) = \frac{1}{2h} \int_{t-h}^{t+h} f(s)\, ds \qquad (f \in L_2).$$

This equality takes on meaning for all $t \in [a,b]$ if we agree to extend every function $f \in L_2$ outside the interval $[a,b]$ by zero.

Obviously S_h is a completely continuous selfadjoint integral operator in L_2 with a bounded kernel

$$\rho_h(t,s) = \begin{cases} 1/2h & \text{for } |t-s| < h, \\ 0 & \text{for } |t-s| > h. \end{cases}$$

The following simple properties of the operator S_h can be verified without difficulty (see for example Ljusternik and Sobolev [1], §10):

1) $|S_h| = 1$ $(h > 0)$.

2) If $f(t)$ $(a \leqq t \leqq b)$ is a continuous function, then $(S_h f)(t)$ $(a \leqq t \leqq b;$ $h > 0)$ is also a continuous function, and as $h \to 0$ the function $(S_h f)(t)$ tends to $f(t)$ uniformly on any interval $[a', b']$, $a < a' < b' < b$.

From 1) and 2) follows:

3) As $h \to 0$ the operator S_h tends strongly to the unit operator, i.e.

$$\lim_{h \to 0} |S_h f - f| = 0 \qquad (f \in L_2).$$

We shall henceforth call a Hilbert-Schmidt kernel $\mathscr{A}(t,s)$ $(a \leqq t, s \leqq b)$ Hermitian nonnegative if the operator corresponding to it by the formula

$$(10.6) \qquad (Af)(t) = \int_a^b \mathscr{A}(t,s) f(s)\, ds$$

is a nonnegative operator, acting in L_2, i.e. if for any $f \in L_2$ we have

$$\int_a^b \int_a^b \mathscr{A}(t,s) f(s) \overline{f(t)}\, ds\, dt \geqq 0.$$

(We remark that this definition does not contradict the definition of the Hermitian nonnegativity of a continuous kernel $\mathscr{A}(t,s)$, given in §10.1.)

With the help of the Steklov operator one can establish the following theorem.

THEOREM 10.1. *In order that a Hermitian nonnegative Hilbert-Schmidt kernel* $\mathscr{A}(t,s)$ $(a \leqq t, s \leqq b)$[13] *generate by formula* (10.6) *a nuclear operator, it is necessary and sufficient that*

$$(10.7) \qquad \overline{\lim_{h \to 0}} \frac{1}{4h^2} \int_a^b \int_a^b [2h - |t-s|]_+ \mathscr{A}(t,s)\, dt\, ds < \infty,$$

where

$$y_+ = \max(y, 0).$$

If the condition (10.7) *is fulfilled, then*

$$(10.8) \quad \operatorname{sp} A = \lim_{h \to 0} \operatorname{sp}(S_h A S_h) = \lim_{h \to 0} \frac{1}{4h^2} \int_a^b \int_a^b [2h - |t-s|]_+ \mathscr{A}(t,s)\, dt\, ds.$$

Thus the theorem asserts in particular that if the limit superior in (10.7)

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[13] Here the interval $[a,b]$ is finite.

is finite, then the ordinary limit of this expression exists and coincides with the trace of the operator A. We shall precede the proof of the theorem with the following remark. If $\mathscr{A}(t,s)$ is a Hilbert-Schmidt kernel, then the kernel

$$\mathscr{A}_h(t,s) = \frac{1}{4h^2}\int_{t-h}^{t+h}\int_{s-h}^{s+h} \mathscr{A}(v,u)\,du\,dv \qquad (a \leqq t,s \leqq b)$$

is continuous. In this formula $\mathscr{A}(t,s)$ is taken to be identically equal to zero outside the square $a \leqq t,s \leqq b$. Obviously

$$\mathscr{A}_h(t,s) = \int_a^b\int_a^b \rho_h(t,u)\,\mathscr{A}(u,v)\,\rho_h(v,s)\,du\,dv,$$

and consequently

$$\int_a^b \mathscr{A}_h(s,s)\,ds = \int_a^b\int_a^b \mathscr{A}(u,v)\int_a^b \rho_h(s,u)\,\rho_h(v,s)\,ds\,du\,dv$$

$$= \frac{1}{4h^2}\int_a^b\int_a^b [2h - |t-s|\,]_+\mathscr{A}(t,s)\,dt\,ds.$$

Thus the condition (10.7) coincides with the condition

$$\overline{\lim_{h\to 0}}\int_a^b \mathscr{A}_h(s,s)\,ds < \infty, \tag{10.9}$$

and formula (10.8) coincides with the formula

$$\operatorname{sp} A = \lim_{h\to 0}\operatorname{sp}(S_hAS_h) = \lim_{h\to 0}\int_a^b \mathscr{A}_h(s,s)\,ds. \tag{10.10}$$

PROOF OF THE THEOREM. The kernel $\mathscr{A}_h(t,s)$ is the kernel of the operator S_hAS_h:

$$(S_hAS_hf)(t) = \int_a^b \mathscr{A}_h(t,s)\,f(s)\,ds \qquad (f \in L_2).$$

It is continuous and Hermitian nonnegative. Consequently by Result 1 the operator S_hAS_h is nuclear and, according to formula (10.5),

$$|S_hAS_h|_1 = \operatorname{sp}(S_hAS_h) = \int_a^b \mathscr{A}_h(s,s)\,ds. \tag{10.11}$$

Since the operator A is completely continuous, by Theorem 6.3 the operator S_hAS_h tends uniformly to A as $h \to 0$. Therefore if (10.9) holds, i.e.

$$\overline{\lim_{h\to 0}}\,|S_hAS_h|_1 < \infty,$$

then according to Theorem 5.2 $A \in \mathfrak{S}_1$.

We now prove the necessity of the hypotheses of the theorem. Let $A \in \mathfrak{S}_1$; then according to Theorem 6.3 the operator S_hAS_h tends to the operator A in the norm of $\mathfrak{S}_1$ as $h \to 0$. Since $\operatorname{sp} X$ is a continuous functional on $\mathfrak{S}_1$, we have

$$\operatorname{sp} A = \lim_{h \to 0} \operatorname{sp}(S_hAS_h),$$

ans so, by (10.11)

$$\operatorname{sp} A = \lim_{h \to 0} \int_a^b \mathscr{A}_h(s,s)\,ds < \infty.$$

The theorem is proved.

The following results are an immediate consequence of this theorem.

COROLLARY 10.1. *If $\mathscr{A}(t,s)$ $(a \leqq t,s \leqq b)$ is a Hermitian nonnegative bounded kernel, then it gives rise to a nuclear operator A.*

Indeed, in this case the condition (10.9) is fulfilled, as

$$\sup_{a \leqq t,s \leqq b;\, h} |\mathscr{A}_h(t,s)| \leqq \sup_{a \leqq t,s \leqq b} |\mathscr{A}(t,s)|. \tag{10.12}$$

Up to now we have been concerned with criteria for the nuclearity of integral operators with Hermitian positive kernels. For arbitrary kernels we can mention the following result.

COROLLARY 10.2. *If the Hilbert-Schmidt kernel $\mathscr{A}(t,s)$ $(a \leqq t,s \leqq b)$ gives rise to a nuclear operator A, then, as before,*

$$\begin{aligned} \operatorname{sp} A &= \lim_{h \to 0} \operatorname{sp}(S_hAS_h) \\ &= \lim_{h \to 0} \frac{1}{4h^2} \int_a^b \int_a^b [2h - |t-s|\,]_+ \mathscr{A}(t,s)\,dt\,ds. \end{aligned} \tag{10.13}$$

If moreover this kernel is bounded, then

$$|\operatorname{sp} A| \leqq (b-a) \sup_{a \leqq t,s \leqq b} |\mathscr{A}(t,s)|, \tag{10.14}$$

and if it is continuous, then

$$\operatorname{sp} A = \int_a^b \mathscr{A}(s,s)\,ds.$$

In fact, the kernel $\mathscr{A}(t,s)$ can be represented in the form of a linear combination of four Hermitian nonnegative kernels, each of which gives rise to a nuclear operator, and consequently (10.13) follows from (10.8).

The bound (10.14) follows from the relation (10.12), which holds for any bounded kernel, and the relation (10.13).

Finally, if the kernel $\mathscr{A}(t,s)$ is continuous, the function $\mathscr{A}_h(s,s)$ tends to the function $\mathscr{A}(s,s)$ uniformly on any interval $[a',b']$, $a<a'<b'<b$, as $h\to 0$; using (10.12) it follows that

$$\overline{\lim_{h\to 0}}\int_a^b \mathscr{A}_h(s,s)\,ds=\int_a^b \mathscr{A}(s,s)\,ds.$$

3. It turns out that the continuity alone of the kernel $\mathscr{A}(t,s)$ over all of the square $a\leqq t,s\leqq b$ does not guarantee the nuclearity of the corresponding operator A. It does not even guarantee that A belong to at least one of the classes $\mathfrak{S}_p$ ($p<2$). This fact was first discovered by T. Carleman [**1**] (cf. N. K. Bari [**3**]). Carleman constructed a continuous periodic function $\Phi(t)=\Phi(t+1)$ with a Fourier expansion

$$\Phi(t)\sim\sum_{k=-\infty}^{\infty}c_k e^{2\pi ikt}, \tag{10.15}$$

in which

$$\sum_{k=-\infty}^{\infty}|c_k|^p=\infty \quad \text{for any } p<2. \tag{10.16}$$

The point is that the sequence $\{c_k\}_{-\infty}^{\infty}$ of Fourier coefficients of such a function forms the complete system of eigenvalues of the normal operator A_Φ in $L_2(0,1)$ with kernel $\mathscr{A}_\Phi(t,s)=\Phi(t-s)$ $(0\leqq t,s\leqq 1)$, and therefore the condition (10.16) shows that $A_\Phi\notin\mathfrak{S}_p$ for any positive $p<2$.

We can supplement this result of Carleman by the remark that the operator A_Φ will be nuclear, if the function $\Phi(t)=\Phi(t+1)$ belongs to the Lipschitz class $\operatorname{Lip}\alpha$ with $\alpha>\frac{1}{2}$; indeed, in this case by a theorem of S. N. Bernšteĭn (cf. N. K. Bari [**3**], Russian p. 209) the series (10.15) is absolutely convergent, and hence

$$\sum_{k=-\infty}^{\infty}|c_k|<\infty.$$

It is interesting to compare this last result with results of Fredholm [**1**] from his classical memoir.

If the kernel $\mathscr{A}(t,s)$ $(a\leqq t,s\leqq b)$ satisfies the condition $\operatorname{Lip}\alpha$ $(0<\alpha\leqq 1)$ in the variable s,

$$|\mathscr{A}(t,s_2)-\mathscr{A}(t,s_1)|\leqq C|s_2-s_1|^\alpha, \tag{10.17}$$

where the constant C does not depend on $t\in[a,b]$, then

$$\sum_{j} |\lambda_j(A)|^p < \infty \quad \text{for} \quad p > \frac{2}{2\alpha + 1}. \tag{10.18}$$

Thus a Hermitian operator A with a kernel satisfying the condition (10.17) will be nuclear as long as $\alpha > \frac{1}{2}$.

Fredholm's assertion remains valid when the numbers $\lambda_j(A)$ in (10.18) are replaced by the numbers $s_j(A)$, and consequently the last conclusion will also be valid for non-Hermitian operators A. This result is essentially contained in a paper by W. F. Stinespring [**1**].

We shall obtain below a sharpening of Fredholm's result for "smooth" kernels.

4. Let $\mathscr{A}(t,s)$ and $\mathscr{B}(t,s)$ $(a \leqq t, s \leqq b)$ be two kernels, satisfying the conditions

$$\begin{gathered} \int_a^b |\mathscr{A}(t,s)|^2 dt < \infty, \\ \int_a^b |\mathscr{B}(t,s)|^2 dt < \infty \quad (a \leqq s \leqq b). \end{gathered} \tag{10.19}$$

Let us agree to write

$$\mathscr{B}(t,s) = \mathscr{A}_{01}(t,s)$$

and say that the kernel $\mathscr{B}(t,s)$ is the derivative in the mean (with respect to s) of the kernel $\mathscr{A}(t,s)$ if

$$\lim_{h \to 0} \int_a^b \left| \mathscr{B}(t,s) - \frac{\mathscr{A}(t,s+h) - \mathscr{A}(t,s)}{h} \right|^2 dt = 0 \quad (a \leqq s \leqq b). \tag{10.20}$$

By virtue of (10.19) the kernels $\mathscr{A}(t,s)$ and $\mathscr{B}(t,s)$ can be treated as vector-functions $X(s)$ and $Y(s)$ $(a \leqq s \leqq b)$ with values in $L_2(a,b)$. For such an interpretation the relation (10.20) means that the vector-function $Y(s)$ is everywhere in $[a,b]$ the strong derivative dX/ds of the vector-function $X(s)$. If $g(s)$ is a numerical continuously differentiable function and $g(a) = g(b) = 0$, then

$$\int_a^b X(s)\, g'(s)\, ds = -\int_a^b \frac{dX}{ds} g(s)\, ds,$$

and the integrals converge in the mean (i.e. in the norm of $L_2(a,b)$). Therefore, under the indicated conditions with regard to $g(s)$ we will have, in the sense of convergence in the mean,

$$\int_a^b \mathscr{A}(t,s)\, g'(s)\, ds = -\int_a^b \mathscr{A}_{01}(t,s)\, g(s)\, ds. \tag{10.21}$$

If the vector-function $X(s) = \mathscr{A}(t,s)$ is strongly differentiable in $[a,b]$, then it is strongly continuous, and consequently its norm $|X(s)|$ is continuous in $[a,b]$. Therefore, in this case, always

$$\int_a^b \int_a^b |\mathscr{A}(t,s)|^2 dt\, ds = \int_a^b |X(s)|^2 ds < \infty,$$

i.e. $\mathscr{A}(t,s)$ is a Hilbert-Schmidt kernel.

It is easily seen that if $\mathscr{A}_{01}(t,s)$ is a Hilbert-Schmidt kernel, then (10.21) will hold in the sense of ordinary Lebesgue integrals almost everywhere in $[a,b]$.

One has the result

3. *If the kernel $\mathscr{A}(t,s)$ of a Hilbert-Schmidt operator A in $L_2(a,b)$ has a derivative in the mean $\mathscr{A}_{01}(t,s)$ which is a Hilbert-Schmidt kernel, then*

$$\sum_{n=1}^{\infty} n^2 s_n^2(A) < \infty, \tag{10.22}$$

and consequently

$$s_n(A) = o(n^{-3/2}) \qquad (n \to \infty) \tag{10.23}$$

and $A \in \mathfrak{S}_p$ for any $p > 2/3$.

To prove this assertion we introduce the operator of integration

$$J_1 f = \int_t^b f(s)\, ds.$$

Since the numbers $s_n^2(J_1)$ $(n = 1, 2, \cdots)$ are the eigenvalues of the operator $J_1^* J_1$, the numbers $\lambda_n = s_n^{-2}(J_1)$ will be the characteristic numbers of the integral equation

$$\phi(t) = \lambda \int_a^t \left(\int_s^b \phi(u)\, du \right) ds,$$

which is equivalent to the boundary value problem

$$\phi'' + \lambda\phi = 0, \quad \phi(a) = 0, \quad \phi'(b) = 0.$$

The sequence $\lambda_n = \pi^2(2n-1)^2/4(b-a)^2$ $(n = 1, 2, \cdots)$ constitutes the complete system of characteristic numbers of this problem, and therefore

$$s_n(J_1) = \frac{b-a}{\pi} \frac{2}{2n-1} \quad (n = 1, 2, \cdots).$$

Let us denote by $\mathfrak{L}_0$ the subspace consisting of all $f \in L_2(a,b)$ which are orthogonal to the function $e(t) \equiv 1$, and by Q the orthoprojector which projects $L_2(a,b)$ onto $\mathfrak{L}_0$. Then $P = I - Q$ will be a one-dimensional orthoprojector:

$$(Pf)(t) = \frac{1}{b-a}\int_a^b f(t)\,dt \qquad (a \leqq t \leqq b).$$

If $f \in \mathfrak{L}_0$, i.e. $Pf = 0$, then $g = -J_1 f$ will be a solution of the system

$$dg/dt = f(t), \quad g(a) = g(b) = 0.$$

Let us consider the operator $A_0 = AQ$. Since $A - A_0 = AP$ is a one-dimensional operator, it follows that

$$(10.24) \qquad s_{n+1}(A_0) \leqq s_n(A) \leqq s_{n-1}(A_0) \qquad (n = 2, 3, \cdots).$$

Now let $f(t) \in C(a,b)$. Since $PQf = 0$, there exists a continuously differentiable function $g(t) = -(J_1 Qf)(t)$ such that

$$dg/dt = (Qf)(t), \quad g(a) = g(b) = 0.$$

Then

$$(A_0 f)(t) = \int_a^b \mathscr{A}(t,s)\frac{dg}{ds}\,ds = -\int_a^b \mathscr{A}_{01}(t,s)\,g(s)\,ds = (BJ_1Qf)(t),$$

where B is a Hilbert-Schmidt operator with kernel $\mathscr{B}(t,s) = \mathscr{A}_{01}(t,s)$. Since $C(a,b)$ is dense in $L_2(a,b)$, we conclude that

$$A_0 = BJ_1Q.$$

According to Corollary II.2.2 and the result II of §2.1, Chapter II,

$$s_{2n}(A_0) \leqq s_{2n-1}(A_0) \leqq s_n(B)s_n(J_1) = \frac{2(b-a)}{\pi(2n-1)}\,s_n(B),$$

and so

$$\sum_n (2n-1)^2\left[s_{2n}^2(A_0) + s_{2n-1}^2(A_0)\right] \leqq \frac{8(b-a)^2}{\pi^2}\sum_{n=1}^{\infty} s_n^2(B) < \infty.$$

Comparing this with inequality (10.24), we arrive at (10.22). We shall clarify the conclusion (10.23) drawn from (10.22) in a moment, in connection with a more general result.

The notion of the first derivative in the mean of a kernel $\mathscr{A}(t,s)$ enables one to introduce the notion of the lth derivative in the mean $\mathscr{A}_{0l}(t,s)$ $(l = 1, 2, \cdots)$.

By the same methods[14] one can obtain the following generalization of the result just proved.

4. *If the kernel* $\mathscr{A}(t,s)$ *of the Hilbert-Schmidt operator* A *in* $L_2(a,b)$

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[14] The basic idea of this method is borrowed from a paper by M. G. Kreĭn [1], in which related results were presented.

has an lth derivative in the mean $\mathscr{A}_{0l}(t,s)$ *which is a Hilbert-Schmidt kernel, then*

$$\sum_{n=1}^{\infty} n^{2l} s_n^2(A) < \infty, \tag{10.25}$$

and consequently

$$s_n(A) = o(n^{-(l+1/2)}) \qquad (n \to \infty) \tag{10.26}$$

and $A \in \mathfrak{S}_p$ *for* $p > 1/(l+\frac{1}{2})$.

Let us clarify the conclusion drawn from (10.25).

We denote by $n(r)$ $(0 < r < \infty)$ the number of numbers $s_n^{-1}(A)$ lying in the segment $[0,r]$. From (10.25) follows

$$\int_0^{\infty} \frac{dn^{2l+1}(r)}{r^2} < \infty.$$

On the other hand,

$$\int_0^R \frac{dn^{2l+1}(r)}{r^2} = \frac{n^{2l+1}(R)}{R^2} + 2\int_0^R \frac{n^{2l+1}}{r^3}\, dr.$$

Since as $R \to \infty$ the left side tends to a finite limit, each of the nonnegative terms (of which the second is monotone) in the right side tends to a finite limit. But then the first term must tend to zero, as otherwise the integral in the second term would diverge. Thus (10.26) is proved.

According to Corollary II.3.2, it follows from (10.26) that

$$\lambda_n(A) = o(n^{-(l+1/2)}). \tag{10.27}$$

For Hermitian kernels $\mathscr{A}(t,s)$, the relation (10.27) was obtained by H. Weyl [1] under stronger requirements with regard to the derivatives $\mathscr{A}_{0k}(t,s)$ $(k = 0, 1, \cdots, l)$, then for $l = 1$ under still stronger requirements for non-Hermitian kernels by S. Mazurkiewicz [1] and, finally, for any integer l and also under stronger requirements than in result 4, A. O. Gel′fond [1, 2] obtained, for any $\epsilon > 0$,

$$\lambda_n(A) = o\left(\frac{1}{n^{l+1/2+\epsilon}}\right).$$

Our considerations are an illustration of a general method which enables one to obtain, for specified conditions on the kernel $\mathscr{A}(t,s)$ of the operator, relations of the type (10.25). This method admits of generalization to the case of integral operators acting in $L_2(Q)$, where Q is some open region of n-dimensional Euclidean space. We also point out that

various tests for the nuclearity of such operators were recently obtained by W. F. Stinespring [1].[15)]

A number of results on the asymptotic behavior of the spectrum of an integral operator can be found in the memoir of Hille-Tamarkin [1], which has already been cited in other connections.

5. It goes without saying that the results presented in this section can be generalized to the case of operators with matrix kernels, acting in spaces $L_2^{(r)}(a,b)$.

We shall present, for example, the appropriate generalizations of result 2 and Theorem 10.1.

Let $\sigma(t) = \| \sigma_{jk}(t) \|_1^r$ $(a \leqq t \leqq b)$ be an arbitrary matrix-function having the following two properties:

a) all of the functions $\sigma_{jk}(t)$ have bounded variation;

b) for any t_1, t_2 $(a \leqq t_1 < t_2 \leqq b)$ the matrix $\Delta\sigma(t) = \sigma(t_2) - \sigma(t_1)$ is Hermitian and nonnegative.

We define, in the space $C^{(r)}(a,b)$ of all continuous vector-functions $f(t) = \{ f_j(t) \}_1^r$, a scalar product

$$(f,g)_\sigma = \int_a^b g^*(s)\, d\sigma(s)\, f(s) \quad (f,g \in C^{(r)}). \tag{10.28}$$

Completing $C^{(r)}(a,b)$ in the norm generated by the scalar product (10.28), we obtain a Hilbert space, denoted by $L_{2,\sigma}^{(r)}(a,b)$.[16)]

It is easily seen that every continuous matrix kernel $\mathscr{A}(t,s)$ (i.e., a matrix kernel with continuous elements) $(a \leqq t, s \leqq b)$ gives rise to a Hilbert-Schmidt operator in $L_{2,\sigma}^{(r)}$, if for any $f \in C^{(r)}(a,b)$ we put

$$(Af)(t) = \int_a^b \mathscr{A}(t,s)\, d\sigma(s)\, f(s). \tag{10.29}$$

A matrix kernel $\mathscr{A}(t,s) = \| a_{jk}(t,s) \|_1^r$ with elements from L_1 over the basic square is said to be Hermitian nonnegative if, for any vector-function $\phi(t)$ $(\in C^{(r)}(a,b))$

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15) The applicability of the method discussed here for obtaining various characteristics of the s-numbers of multi-dimensional linear operators which increase smoothness, was considered in recent work of V. I. Paraska [1]. In this work a comparison is given of the results obtained in this way with the results of S. Agmon [1]. Apparently Agmon did not know of M. G. Kreĭn's work [4].

16) One can find a functional-theoretic characterization of the elements of the space $L_{2,\sigma}^{(r)}(a,b)$, given by I. S. Kac, in the book by Ahiezer and Glazman [1] (§72, Chapter VI of 1950 edition; §86, Chapter VI of 1966 edition).

(10.30)
$$\int_a^b\int_a^b \phi^*(t)\mathscr{A}(t,s)\phi(s)\,dt\,ds \geqq 0.$$

For a continuous kernel $\mathscr{A}(t,s)$ condition (10.30) is equivalent to

$$\sum_{j,k=1}^{n} \xi_j^*\mathscr{A}(t_j,t_k)\,\xi_k \geqq 0,$$

where the t_j ($j=1,2,\cdots,n$; $n=1,2,\cdots$) are any points of the interval $[a,b]$, and the ξ_j ($j=1,2,\cdots,n$) are any vectors from E_r.

The Hilbert-Schmidt-Mercer theory can be carried over to the operator A, generated in $L_{2,\sigma}^{(r)}$ by a Hermitian nonnegative continuous kernel $\mathscr{A}(t,s)$ according to the formula (10.29).

It follows that such an operator is always nuclear and that its trace can be calculated from the formula [17]

$$\mathrm{sp}\,A = \int_a^b \mathbf{sp}[\mathscr{A}(s,s)\,d\sigma(s)].$$

The matrix analog of Theorem 10.1 will be the following result.

THEOREM 10.1′. *In order that a Hermitian nonnegative Hilbert-Schmidt matrix kernel* $\mathscr{A}(t,s) = \|a_{jk}(t,s)\|_1^r$ $(a \leqq t, s \leqq b)$ *generate, by the formula*

$$(Af)(t) = \int_a^b \mathscr{A}(t,s)f(s)\,ds,$$

a nuclear operator in $L_2^{(r)}(a,b)$, *it is necessary and sufficient that*

(10.31)
$$\overline{\lim_{h\to0}}\int_a^b \mathbf{sp}\,\mathscr{A}_h(s,s)\,ds < \infty,$$

where

$$\mathscr{A}_h(t,s) = \frac{1}{4h^2}\int_{t-h}^{t+h}\int_{s-h}^{s+h}\mathscr{A}(u,v)\,du\,dv.$$

If the condition (10.31) *is fulfilled, then*

(10.32)
$$\mathrm{sp}\,A = \lim_{h\to0}\int_a^b \mathbf{sp}\,\mathscr{A}_h(s,s)\,ds.$$

We remark that, as in the scalar case, condition (10.31) and formula (10.32) can be written in another form, if one notes that

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[17] In this and succeeding formulas the trace of a finite matrix will for clarity be denoted by **sp**.

$$\int_a^b \mathrm{sp}\,\mathscr{A}_h(s,s)\,ds = \frac{1}{4h^2}\int_a^b\int_a^b |2h - |t-s||_+ \,\mathrm{sp}\,\mathscr{A}(t,s)\,dt\,ds.$$

Let us remark finally that if it is known in advance that A is nuclear, then its trace can be calculated from the formula (10.32). If moreover the kernel $\mathscr{A}(t,s)$ is continuous, then

$$\mathrm{sp}\,A = \int_a^b \mathrm{sp}\,\mathscr{A}(s,s)\,ds.$$

§11. Functions adjoint to s.n. functions

1. Let $\Phi(\xi)$ be some s.n. function and $\eta = \{\eta_j\} \in \hat{k}$ any fixed vector. We consider the function

$$F(\xi) = \frac{1}{\Phi(\xi)}\sum_j \xi_j\eta_j \qquad (\xi \in \hat{k};\ \xi \neq 0)$$

and put

$$F(0) = 0. \tag{11.1}$$

It is obvious that

$$F(\xi) \leqq \sum_j \frac{\xi_j}{\xi_1}\eta_j \leqq \sum_j \eta_j \qquad (\xi \in \hat{k})$$

and that

$$F(\alpha\xi) = F(\xi) \qquad (\xi \in \hat{k})$$

for any positive α. Moreover, if ν is the largest index corresponding to a nonzero component of the vector η, then for any vector $\xi = \{\xi_j\} \in \hat{k}$ we have

$$F(\xi) \leqq F(\xi_1, \xi_2, \cdots, \xi_\nu, 0, 0, \cdots).$$

It follows from the enumerated properties of the function $F(\xi)$ that it is bounded and assumes its supremum. Thus the function

$$\Phi^*(\eta) = \max_{\xi\in\hat{k}}\left[\frac{1}{\Phi(\xi)}\sum_j \eta_j\xi_j\right] \tag{11.2}$$

has meaning for all vectors $\eta \in \hat{k}$. We shall call this function the *adjoint* of the function $\Phi(\xi)$.

THEOREM 11.1. *A function $\Phi^*(\eta)$ $(\eta \in \hat{k})$ which is the adjoint of some s. n. function $\Phi(\xi)$ is itself an s. n. function. The adjoint of the function $\Phi^*(\eta)$ is the function $\Phi(\xi)$.*

PROOF. Obviously, the function $\Phi^*(\eta)$ defined by (11.1) has properties I') - III') of an s.n. function (see §3). Since

$$\frac{1}{\Phi(\xi)}\sum_j \eta_j\xi_j = \frac{1}{\Phi(\xi)}\sum_j \left(\sum_{m=1}^{j}\eta_m\right)(\xi_j - \xi_{j+1}),$$

and $\xi_j \geqq \xi_{j+1}$, the function $\Phi^*(\eta)$ also has the property V'). Finally, since $\xi_1 \leqq \Phi(\xi)$ and $\Phi(1,0,0,\cdots) = 1$, we obtain

$$\Phi^*(1,0,0,\cdots) = 1.$$

Consequently $\Phi^*(\eta)$ is an s. n. function.

Let us now consider the n-dimensional vector space $\hat{C}_n$, consisting of all sequences of *complex numbers* of the form $\xi = \{\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots\}$. The formula

$$(11.3) \qquad |\xi| = \Phi(\xi^*) \qquad (\xi \in \hat{C}_n)$$

defines a norm[18] in the space $\hat{C}_n$. In fact,

$$|\xi| > 0 \qquad (\xi \in \hat{C}_n;\ \xi \neq 0);$$

for any complex λ,

$$|\lambda\xi| = |\lambda|\,|\xi| \qquad (\xi \in \hat{C}_n).$$

We prove the triangle inequality. Let $\xi, \eta \in \hat{C}_n$ and $\zeta = \xi + \eta$. Then obviously

$$\sum_{j=1}^{m}\zeta_j^* \leqq \sum_{j=1}^{m}\xi_j^* + \sum_{j=1}^{m}\eta_j^* \qquad (m = 1, 2, \cdots, n),$$

and so

$$(11.4) \qquad \Phi(\zeta^*) \leqq \Phi(\xi^* + \eta^*).$$

Since

$$(11.5) \qquad \Phi(\xi^* + \eta^*) \leqq \Phi(\xi^*) + \Phi(\eta^*),$$

it follows from (11.4) and (11.5) that

$$|\xi + \eta| \leqq |\xi| + |\eta|.$$

Thus the space $\hat{C}_n$ with the norm (11.3) is a Banach space.

Let

[18] We recall that the vector ξ^* consists of the moduli of the components of the vector ξ, arranged in decreasing order.

$$f(\xi) = \sum_{j=1}^{n} f_j \xi_j \qquad (\xi \in \hat{c}_n)$$

be an arbitrary linear functional from $\hat{c}_n^*$. Then

$$|f| = \max_{\xi \in \hat{c}_n} \left[\frac{1}{\Phi(\xi^*)} \left| \sum_{j=1}^{n} f_j \xi_j \right| \right].$$

It is easily seen that

$$\max_{\xi \in \hat{c}_n} \left[\frac{1}{\Phi(\xi^*)} \left| \sum_{j=1}^{n} f_j \xi_j \right| \right] = \max_{\xi \in \hat{c}_n} \left[\frac{1}{\Phi(\xi^*)} \sum_{j=1}^{n} f_j^* \xi_j^* \right]$$

Moreover

$$\max_{\xi \in \hat{c}_n} \left[\frac{1}{\Phi(\xi^*)} \sum_{j=1}^{n} f_j^* \xi_j^* \right] = \max_{\xi \in \hat{k}} \left[\frac{1}{\Phi(\xi)} \sum_{j=1}^{n} f_j^* \xi_j \right].$$

Consequently

$$|f| = \Phi^*(f^*). \tag{11.6}$$

Since any finite-dimensional Banach space is reflexive, it follows from (11.3) and (11.6) that

$$\Phi^{**}(\xi^*) = \Phi(\xi^*) \qquad (\xi \in \hat{c}_n).$$

The theorem is proved.

2. The following remark will be useful in the sequel.

Let $\Phi(\xi)$ $(\xi \in \hat{k})$ be some s. n. function. If the function $\Psi(\xi)$ $(\xi \in \hat{k})$ satisfies the condition

$$\sum_j \xi_j \eta_j \leqq \Phi(\xi)\, \Psi(\eta) \qquad (\xi, \eta \in \hat{k}), \tag{11.7}$$

and for any vector ξ (or η) from $\hat{k}$ one can find a vector η (or ξ) from $\hat{k}$ for which equality holds in (11.7), then

$$\Phi^*(\eta) = \Psi(\eta) \qquad (\eta \in \hat{k}).$$

As an example, let us consider the s. n. function

$$\Phi_p(\xi) = \left(\sum_j \xi_j^p \right)^{1/p} \qquad (1 \leqq p \leqq \infty; \xi \in \hat{k}).$$

According to Hölder's inequality

$$\sum_j \eta_j \xi_j \leqq \Phi_p(\xi)\, \Phi_q(\eta) \qquad (p^{-1} + q^{-1} = 1; \xi, \eta \in \hat{k}), \tag{11.8}$$

and for any fixed $\xi \in \hat{k}$ equality holds in (11.8) if and only if

$$\eta_j = c\xi_j^{p-1} \qquad (j = 1, 2, \cdots),$$

where c is an arbitrary positive constant; and for fixed $\eta \in \hat{k}$ equality holds in (11.8) if and only if

$$\xi_j = c\eta_j^{q-1} \qquad (j = 1, 2, \cdots).$$

Thus

$$\Phi_p^*(\eta) = \Phi_q(\eta) \qquad (p^{-1} + q^{-1} = 1).$$

In particular, the minimal and maximal s.n. functions are the adjoints of each other. Therefore when an s. n. function $\Phi(\xi)$ is equivalent to the maximal (minimal) one, its adjoint is equivalent to the minimal (maximal) one.

§12. Symmetrically-normed ideals adjoint to separable symmetrically-normed ideals

1. As is well known, in functional analysis $\boldsymbol{c}_0$, $\boldsymbol{l}_p$ ($1 \leq p < \infty$) and $\boldsymbol{m}$ denote the Banach spaces of sequences of numbers $\xi = \{\xi_j\}_1^\infty$ satisfying the respective conditions:

1) $$\lim \xi_j = 0, |\xi|_c = \max_j |\xi_j|;$$

2) $$(|\xi|_p)^p = \sum_j |\xi_j|^p < \infty,$$

3) $$|\xi|_m = \sup_j |\xi_j| < \infty.$$

The following relations exist among these spaces: the space adjoint to $\boldsymbol{l}_1$ is equivalent to $\boldsymbol{m}$, the space adjoint to $\boldsymbol{l}_p$ ($1 < p < \infty$) is equivalent to $\boldsymbol{l}_q$ ($p^{-1} + q^{-1} = 1$) and the space $\boldsymbol{l}_1$ is equivalent to the space adjoint to $\boldsymbol{c}_0$.

More precisely, the first of these assertions means the following.

The general form of a continuous linear functional $f(\xi)$ on the space $\boldsymbol{l}_1$ is given by the formula

$$f(\xi) = \sum_{j=1}^{\infty} f_j \xi_j,$$

where $f = \{f_j\}_1^\infty$ is any vector from $\boldsymbol{m}$, and

$$|f| = |f|_{l_1^*} = \sup_{\xi \in l_1} [|f(\xi)| / |\xi|_{l_1}] = |f|_{\boldsymbol{m}}.$$

The second and third assertions are analogous.

To these results there correspond, in a certain sense, theorems to be presented in this section on the spaces adjoint to $\mathfrak{S}_p$ $(1 \leqq p \leqq \infty)$. In this analogy the roles of the spaces $\boldsymbol{c}_0$, $\boldsymbol{l}_p$ $(1 \leqq p \leqq \infty)$ and $\boldsymbol{m}$ are played, respectively, by the spaces $\mathfrak{S}_\infty$, $\mathfrak{S}_p$ $(1 \leqq p \leqq \infty)$ and the space $\mathfrak{R}$ of all bounded linear operators. All of these results will be obtained as corollaries of general theorems on the spaces adjoint to separable s.n. ideals.

2. THEOREM 12.1. *The general form of a continuous linear functional $F(X)$ on the Banach space $\mathfrak{S}_1$ is given by the formula*

$$F(X) = \mathrm{sp}(AX),$$

where A is any operator from $\mathfrak{R}$, and

$$|F| = \sup_{X \in \mathfrak{S}_1} \frac{|\mathrm{sp}(AX)|}{|X|_1} = |A|.$$

Thus the space $\mathfrak{R}$ is isometric to the space adjoint to $\mathfrak{S}_1$.

PROOF. In fact, let A be any bounded linear operator. Then the formula

$$F(X) = \mathrm{sp}(AX) \qquad (X \in \mathfrak{S}_1)$$

defines a linear functional. This functional is continuous, since by Theorem 8.5

$$|\mathrm{sp}(AX)| \leqq |AX|_1 \leqq |A|\,|X|_1. \tag{12.1}$$

It follows from (12.1) that $|F| \leqq |A|$. To prove that the equal sign holds in the last relation, we consider a sequence of unit vectors ϕ_n such that $\lim_{n\to\infty}|A\phi_n| = |A|$. We form the one-dimensional operators

$$X_n = \frac{1}{|A\phi_n|}(\cdot, A\phi_n)\,\phi_n \qquad (n = 1, 2, \cdots).$$

It is obvious that $|X_n|_1 = 1$ and $F(X_n) = \mathrm{sp}(AX_n) = |A\phi_n|$ $(n = 1, 2, \cdots)$. Thus $|F| = |A|$.

Conversely, let $F(X)$ be any continuous linear functional defined on $\mathfrak{S}_1$. We construct the form

$$B(f,g) = F(X_{f,g}),$$

where $X_{f,g} = (\cdot, g)f$, and f and g are arbitrary vectors from $\mathfrak{H}$.

It is easily seen that $B(f,g)$ is a bilinear form. Moreover, from the relation

$$|F(X_{f,g})| \leqq |F| \, |X_{f,g}|_1 = |F| \, |f| \, |g| \qquad (f,g \in \mathfrak{H})$$

follows

$$|B(f,g)| \leqq |F| \, |f| \, |g| \qquad (f,g \in \mathfrak{H}).$$

This means that the form $B(f,g)$ is bounded. Therefore one can find a bounded linear operator A such that

$$B(f,g) = (Af,g).$$

Thus for any operator $X_{f,g}$

$$F(X_{f,g}) = (Af,g) = \operatorname{sp}(AX_{f,g}). \tag{12.2}$$

Let X be an arbitrary operator from $\mathfrak{S}_1$; then X can be expanded in a convergent in $\mathfrak{S}_1$-norm (see Lemma 6.1) Schmidt series

$$X = \sum_j s_j(X)(\cdot, \phi_j)\psi_j.$$

By the linearity and continuity of the functional $F(X)$, we have

$$F(X) = \sum_j s_j(X) \, F(X_{\psi_j,\phi_j});$$

taking (12.2) into account, we obtain

$$F(X) = \sum_j s_j(X)(A\psi_j, \phi_j) = \operatorname{sp}(AX).$$

The theorem is proved.

3. THEOREM 12.2. *Let $\Phi(\xi)$ be an arbitrary s. n. function, not equivalent to the maximal one. Then the general form of a continuous linear functional $F(X)$ on the separable space $\mathfrak{S}_\Phi^{(0)}$ is given by the formula*

$$F(X) = \operatorname{sp}(AX), \tag{12.3}$$

where A is an arbitrary operator from $\mathfrak{S}_{\Phi^*}$, and

$$|F| = \sup_{X \in \mathfrak{S}_\Phi^{(0)}} \frac{|\operatorname{sp}(AX)|}{|X|_\Phi} = |A|_{\Phi^*}.$$

Thus the space $\mathfrak{S}_{\Phi^*}$ is isometric with the space adjoint to the space $\mathfrak{S}_\Phi^{(0)}$. In particular, if both functions $\Phi(\xi)$ and $\Phi^*(\xi)$ are mononormalizing, the space $\mathfrak{S}_\Phi$ is reflexive.

PROOF. Just as in the proof of Theorem 12.1, we can establish that any operator $A \in \mathfrak{S}_{\Phi^*}$ generates, by formula (12.3), a continuous linear functional F, and

$$|F| \leq |A|_{\Phi^*}. \tag{12.4}$$

We shall show that the equal sign holds in (12.4). To do this, starting from the Schmidt expansion

$$A = \sum_j s_j(A)(\cdot, \phi_j)\psi_j,$$

we form the operators

$$K_n = \sum_{j=1}^{n} \xi_j^{(n)}(\cdot, \psi_j)\phi_j \qquad (n = 1, 2, \cdots),$$

where the vectors $\xi^{(n)} = \{\xi_j^{(n)}\}_1^n$ are chosen so that $\Phi(\xi^{(n)}) = 1$ and

$$\sum_{j=1}^{n} s_j(A)\xi_j^{(n)} = \Phi^*(s_1(A), s_2(A), \cdots, s_n(A), 0, 0, \cdots) = |A_n|_{\Phi^*}. \tag{12.5}$$

Here A_n denotes the nth partial Schmidt series of the operator A. Since

$$AK_n = \sum_{j=1}^{n} s_j(A)\xi_j^{(n)}(\cdot, \psi_j)\psi_j,$$

it follows that

$$F(K_n) = \mathrm{sp}(AK_n) = \sum_{j=1}^{n} s_j(A)\xi_j^{(n)},$$

and so, by (12.5),

$$F(K_n) = |A_n|_{\Phi^*}.$$

Thus

$$\lim_{n\to\infty} F(K_n) = |A|_{\Phi^*}.$$

Bearing in mind that $|K_n|_\Phi = \Phi(\xi^{(n)}) = 1$ $(n = 1, 2, \cdots)$, we conclude that

$$|F| = |A|_{\Phi^*}.$$

Now let $F(X)$ be any continuous linear functional on $\mathfrak{S}_\Phi^{(0)}$; we shall show that it admits the representation (12.3). The functional $F(X)$ is at the same time a continuous linear functional on $\mathfrak{S}_1$, and thus by Theorem 12.1 there exists a bounded linear operator A such that for all $X \in \mathfrak{S}_1$,

$$F(X) = \mathrm{sp}(AX). \tag{12.6}$$

Let us denote by P_n $(n = 1, 2, \cdots)$ a monotone increasing sequence of finite-dimensional orthoprojectors which tends strongly to the unit

operator. We form the functionals

$$F_n(X) = F(P_n X P_n).$$

The functionals $F_n(X)$ are continuous, since

$$|F_n(X)| = |F(P_n X P_n)| \leqq |F|\,|X|_\Phi. \tag{12.7}$$

It follows from (12.6) that

$$F_n(X) = \mathrm{sp}(AP_n X P_n) = \mathrm{sp}(P_n A P_n X).$$

Since the operators $P_n A P_n$ are finite-dimensional, and consequently belong to $\mathfrak{S}_{\Phi^*}$, according to the part of the theorem which has already been proved and (12.7) we have

$$|F_n| = |P_n A P_n|_{\Phi^*} \leqq |F| \qquad (n = 1, 2, \cdots). \tag{12.8}$$

By hypothesis the function $\Phi(\xi)$ is not equivalent to the maximal one, and consequently the function $\Phi^*(\xi)$ is not equivalent to the minimal one. By Theorem 5.2 it follows from (12.8) that $A \in \mathfrak{S}_{\Phi^*}$. The theorem is proved.

4. The mononormalizing function

$$\Phi_p(\xi) = \left(\sum_j \xi_j^p\right)^{1/p}$$

for $p \leqq \infty$ $(p > 1)$ is not equivalent to the maximal one. As has already been mentioned, $\Phi_p^*(\xi) = \Phi_q(\xi)$ $(p^{-1} + q^{-1} = 1)$. Thus from the theorems just proved we get

THEOREM 12.3. *The general form of a bounded linear functional $F(X)$ on $\mathfrak{S}_p$ $(1 < p \leqq \infty)$ is given by the formula*

$$F(X) = \mathrm{sp}(AX),$$

where A is an operator from $\mathfrak{S}_q$ $(p^{-1} + q^{-1} = 1)$, and

$$|F| = \sup_{X \in \mathfrak{S}_p} \frac{|\mathrm{sp}(AX)|}{|X|_p} = |A|_q.$$

Thus the space adjoint to $\mathfrak{S}_p$ is isometric with the space $\mathfrak{S}_q$ $(p^{-1} + q^{-1} = 1)$, and consequently every space $\mathfrak{S}_p$ $(1 < p < \infty)$ is reflexive.

5. Theorem 12.2 enables us to formulate, as a supplement to Theorem 5.2, still another criterion, to be used further on, for a bounded operator to belong to a space $\mathfrak{S}_\Phi$.

LEMMA 12.1. *Let $\Phi(\xi)$ be an arbitrary s.n. function not equivalent to*

the minimal one. In order that $A \in \mathfrak{S}_\Phi$ *it is necessary and sufficient that*

$$\sup_{K \in \mathfrak{R}} \frac{|\mathrm{sp}(AK)|}{|K|_{\Phi^*}} < \infty, \tag{12.9}$$

where $\mathfrak{R}$, *as before, is the set of all finite-dimensional operators.*

PROOF. The necessity of the condition (12.9) is obvious; we shall prove its sufficiency.

If the condition (12.9) is fulfilled, the functional $F(K) = \mathrm{sp}(AK)$ will be continuous on $\mathfrak{R}$, considered as a part of $\mathfrak{S}_{\Phi^*}^{(0)}$. Since $\mathfrak{R}$ is dense in $\mathfrak{S}_{\Phi^*}^{(0)}$, the functional $F(K)$ can be extended to all of $\mathfrak{S}_{\Phi^*}^{(0)}$ while preserving its linearity and continuity. Following this, by Theorem 12.2 it will admit a representation of the form

$$F(X) = \mathrm{sp}(BX) \qquad (X \in \mathfrak{S}_{\Phi^*}^{(0)}),$$

where B is some operator from $\mathfrak{S}_\Phi$. Thus for $K \in \mathfrak{R}$ we will have

$$\mathrm{sp}(AK) = \mathrm{sp}(BK).$$

Writing this equality for an arbitrary one-dimensional operator $K = (\cdot, \phi)\psi$ $(\phi, \psi \in \mathfrak{H})$, we obtain

$$(A\phi, \psi) = (B\phi, \psi) \qquad (\phi, \psi \in \mathfrak{H}).$$

Hence we conclude that $A = B \in \mathfrak{S}_\Phi$. The lemma is proved.

6. Using the definition and properties of the s-numbers of an arbitrary bounded linear operator (see §7, Chapter II), one can generalize Theorem 12.1 to any s.n. ideal $\mathfrak{S}_\Phi^{(0)}$, defined by an arbitrary s.n. function equivalent to the maximal one. To formulate the theorem we need the notion of a symmetric norm in $\mathfrak{R}$.

If the functional $|A|_\mathfrak{S}$ is defined on all of $\mathfrak{R}$ and has, on the ring $\mathfrak{R}$, all of the properties 1)—5) of a symmetric norm, we shall call $|A|_\mathfrak{S}$ a *symmetric norm on* $\mathfrak{R}$. Since

$$|A| \leqq |A|_\mathfrak{S} \leqq |I|_\mathfrak{S} |A|,$$

every symmetric norm on $\mathfrak{R}$ is topologically equivalent to the ordinary operator norm.

It is easy to show that for every symmetric norm on $\mathfrak{R}$

$$|A| \leqq |A|_\mathfrak{S} \leqq \sum_j s_j(A) \qquad (A \in \mathfrak{R}). \tag{12.10}$$

Let $\Phi(\xi)$ be an arbitrary s.n. function, equivalent to the minimal one. Then the formula

(12.11) $$|A|_\Phi = \lim_{n\to\infty} \Phi(s_1(A), s_2(A), \cdots, s_n(A), 0, 0, \cdots)$$

defines a symmetric norm on $\mathfrak{R}$.

In fact, it follows from the relation

$$\sup_n \Phi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) < \infty$$

that

$$|A|_\Phi < \infty$$

for any $A \in \mathfrak{R}$. It can be verified without difficulty that the norm $|A|_\Phi$ satisfies all of the conditions 1)—5) of a symmetric norm on $\mathfrak{R}$.

If $\Phi(\xi)$ is an arbitrary s.n. function, equivalent to the minimal one, then we shall denote by $\mathfrak{R}_\Phi$ the normed ring of all bounded operators in which the norm is defined by (12.11).

THEOREM 12.4. *Let $\Phi(\xi)$ be an arbitrary s.n. function equivalent to the maximal one. Then the general form of a continuous linear functional $F(X)$ on the separable space $\mathfrak{S}_\Phi$ is given by the formula*

(12.12) $$F(X) = \mathrm{sp}(AX),$$

where A is an arbitrary operator from $\mathfrak{R}$, and

$$|F| = \sup_{X\in\mathfrak{S}_\Phi} \frac{|\mathrm{sp}(AX)|}{|X|_\Phi} = |A|_{\Phi^*}.$$

Thus the space $\mathfrak{R}_{\Phi^*}$ is isometric with the space adjoint to the s.n. ideal $\mathfrak{S}_\Phi$.

PROOF. Since the spaces $\mathfrak{S}_\Phi$ and $\mathfrak{S}_1$ consist of the same operators, and the norms in them are topologically equivalent, the formula

(12.13) $$F(X) = \mathrm{sp}(AX) \qquad (X \in \mathfrak{S}_\Phi),$$

where $A \in \mathfrak{R}$, gives the general form of a continuous linear functional on $\mathfrak{S}_\Phi$.

Let K be an arbitrary finite-dimensional operator; then

$$|\mathrm{sp}(AK)| \leq |AK|_1 = \sum_j s_j(AK),$$

and consequently, by the inequality (7.9),

(12.14) $$|\mathrm{sp}(AK)| \leq \sum_j s_j(A) s_j(K).$$

The function $\Phi^*(\xi)$ is equivalent to the minimal one, and consequently the quantity $|A|_{\Phi^*}$ is meaningful. By the definition of an s.n. function, it follows from (12.14) that

$$|\operatorname{sp}(AK)| \leqq |A|_{\Phi^*} \cdot |K|_{\Phi} \qquad (K \in \mathfrak{K}). \tag{12.15}$$

This shows that

$$|F| \leqq |A|_{\Phi^*}. \tag{12.16}$$

We shall prove that the equal sign holds in (12.16).

Let the polar representation of the operator A be $A = HU$. We denote by p ($\leqq \infty$) the smallest number for which $s_{p+1}(A) = s_\infty(A)$, and by ϕ_j ($j = 1, 2, \cdots, p$) an orthonormal system of eigenvectors of the operator H, corresponding to the eigenvalues $s_j(A)$ ($j = 1, 2, \cdots, p$). With every positive number ϵ we associate the subspace

$$\mathfrak{N}_\epsilon = E(s_\infty(A) + 0)\mathfrak{H} \ominus E(s_\infty(A) - \epsilon)\mathfrak{H},$$

where $E(\lambda)$ is the spectral function of the operator H. If the number p is finite, then the subspace $\mathfrak{N}_\epsilon$ is infinite-dimensional. In this case we denote by ϕ_j ($j = p + 1, p + 2, \cdots$) an arbitrary orthonormal system of vectors from $\mathfrak{N}_\epsilon$.

Thus the system $\{\phi_j\}$ is always an infinite orthonormal system. We form the operators H_n, putting

$$H_n = \sum_{j=1}^{n} s_j(A)(\cdot, \phi_j)\phi_j \qquad (n = 1, 2, \cdots).$$

It is easily seen that if P_n denotes the orthoprojector onto the subspace $\mathfrak{M}_{\epsilon,n}$ with basis $\{\phi_j\}_1^n$, then

$$|P_n H P_n - H_n| < \epsilon \qquad (n = 1, 2, \cdots). \tag{12.17}$$

We also form the finite-dimensional operators L_n, putting

$$L_n = \sum_{j=1}^{n} \xi_j^{(n)}(\cdot, \phi_j)\phi_j \qquad (n = 1, 2, \cdots),$$

where the vectors $\xi^{(n)} = \{\xi_j^{(n)}\}_1^n$ are chosen so that $\Phi(\xi^{(n)}) = 1$ and

$$\sum_{j=1}^{n} s_j(A)\xi_j^{(n)} = \Phi^*(s_1(A), \cdots, s_n(A), 0, 0, \cdots) = |H_n|_{\Phi^*} \qquad (n = 1, 2, \cdots). \tag{12.18}$$

It is obvious that $|L_n|_\Phi = 1$ ($n = 1, 2, \cdots$).

For the operators $K_n = U^* L_n$ we will have

$$|K_n|_\Phi = 1 \text{ and } \operatorname{sp}(AK_n) = \operatorname{sp}(HUU^*L_n) = \operatorname{sp}(HL_n) \qquad (n = 1, 2, \cdots).$$

Since

$$\operatorname{sp}(HL_n) = \operatorname{sp}(P_n H P_n L_n),$$

it follows that

$$\mathrm{sp}(HL_n) = \mathrm{sp}(H_n L_n) - \mathrm{sp}((H_n - P_n H P_n) L_n),$$

and consequently

$$\mathrm{sp}(HL_n) \geqq \mathrm{sp}(H_n L_n) - |H_n - P_n H P_n| \, |L_n|_1. \tag{12.19}$$

By hypothesis the function $\Phi(\xi)$ is equivalent to the maximal one, i.e.

$$C = \sup_{\xi \in \boldsymbol{k}} \left[\frac{1}{\Phi(\xi)} \sum_j \xi_j \right] < \infty .$$

It follows that

$$|L_n|_1 \leqq C |L_n|_\Phi = C. \tag{12.20}$$

From (12.17), (12.19) and (12.20) follows the relation

$$\mathrm{sp}(HL_n) \geqq \mathrm{sp}(H_n L_n) - C\epsilon. \tag{12.21}$$

Bearing in mind that

$$H_n L_n = \sum_{j=1}^{n} s_j(A) \, \xi_j^{(n)} (\cdot, \phi_j) \phi_j,$$

we obtain

$$\mathrm{sp}(H_n L_n) = \sum_{j=1}^{n} s_j(A) \, \xi_j^{(n)}.$$

Thus, by (12.18) and (12.21),

$$\mathrm{sp}(AK_n) = \mathrm{sp}(HL_n) \geqq |H_n|_{\Phi^*} - C\epsilon$$

and

$$\sup_n \mathrm{sp}(AK_n) \geqq |A|_{\Phi^*} - C\epsilon.$$

The theorem is proved.

§13. The three lines theorem for operator-functions which wander in $\mathfrak{S}_p$ spaces

The following result, which bears the name "three lines theorem," is well known in the theory of functions:

Let $f(z)$ be a function, holomorphic in the strip $a \leqq \mathrm{Re}\, z \leqq b$. If

$$|f(a+iy)| \leqq C_1, \quad |f(b+iy)| \leqq C_2 \qquad (-\infty < y < \infty),$$

and

$$\ln |f(z)| \leqq N \exp(k |\mathrm{Im}\, z|) \qquad (a < \mathrm{Re}\, z < b),$$

where $0 \leqq k < \pi/(b-a)$, *then for any* x $(a < x < b)$

$$|f(x+iy)| \leqq C_1^{1-t_x} C_2^{t_x} \qquad (-\infty < y < \infty),$$

where $t_x = (x-a)/(b-a)$.

This result can be generalized without difficulty to vector-functions with values in an arbitrary Banach space (cf. Dunford and Schwartz [**1**]).

More subtle arguments are required for the proof of a result which we have called "the three lines theorem for wandering operator-functions."

We precede the formulation of the theorem by two definitions. Let H be a nonnegative operator from $\mathfrak{S}_\infty$ and

$$H = \sum_j \lambda_j(\cdot, \phi_j)\phi_j$$

its spectral representation. Then we agree to understand by H^z $(\mathrm{Re}\, z \geqq 0)$ the operator defined by

$$H^z = \sum_j \lambda_j^z(\cdot, \phi_j)\phi_j \qquad (\lambda^z = e^{z \ln \lambda},\ -\infty < \ln \lambda < \infty).$$

Obviously H^z $(\mathrm{Re}\, z > 0)$ is a normal operator.

An operator-function $T_z \in \mathfrak{R}$ $(z \in G)$ is said to be holomorphic in the region G, if for any vectors $\phi, \psi \in \mathfrak{H}$ the scalar function $(T_z\phi, \psi)$ is holomorphic in the region G.[19)]

THEOREM 13.1. *Let* $T_z \in \mathfrak{R}$ *be an operator-function holomorphic in the strip* $a \leqq \mathrm{Re}\, z \leqq b$. *Suppose that the values of* T_z *belong, on the line* $z = a + iy$ $(-\infty < y < \infty)$, *to the space* $\mathfrak{S}_{r_1}$ $(1 \leqq r_1 < \infty)$, *and on the line* $z = b + iy$ $(-\infty < y < \infty;\ a < b)$, *to the space* $\mathfrak{S}_{r_2}$ $(r_1 < r_2 \leqq \infty)$. *If*

$$|T_{a+yi}|_{r_1} \leqq C_1, \quad |T_{b+yi}|_{r_2} \leqq C_2 \qquad (-\infty < y < \infty),$$

and if for every pair of vectors $f, g \in \mathfrak{H}$

$$\ln|(T_z f, g)| \leqq N_{f,g} \exp(k_{f,g}|\mathrm{Im}\, z|) \qquad (a < \mathrm{Re}\, z < b),$$

where $0 \leqq k_{f,g} < \pi/(b-a)$, *then on every intermediate line* $z = x + iy$ $(a < x < b;\ -\infty < y < \infty)$ *the values of the operator-function* T_z *belong to* $\mathfrak{S}_r$, *where*

$$r^{-1} = r_1^{-1} + t_x(r_2^{-1} - r_1^{-1}), \qquad t_x = \frac{x-a}{b-a},$$

and

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19) This "weak" holomorphness is equivalent to the "strong" holomorphness defined in Chapter I.

$$(13.1) \qquad |T_{x+iy}|_r \leqq C_1^{1-t_x} C_2^{t_x} \qquad (-\infty < y < \infty).$$

PROOF. The theorem will be proved if the relation (13.1) is proved for the special value $y = 0$, i.e.

$$(13.2) \qquad |T_x|_r \leqq C_1^{1-t_x} C_2^{t_x}$$

In fact, the operator-function $T'_z = T_{z+ih}$ $(-\infty < h < \infty)$ obviously satisfies all the hypotheses of the theorem, and the relation (13.2) says the same for it as does (13.1) for the original operator-function.

Let K be an arbitrary finite-dimensional operator, and $K = UH$ its polar representation. Assuming $|K|_{r'} = 1$, $(r')^{-1} + r^{-1} = 1$, let us consider the function

$$f(z) = \mathrm{sp}[T_z UH^{\alpha+\beta z}] \qquad (a \leqq \mathrm{Re}\, z \leqq b),$$

where

$$\alpha + \beta z = r'\left(\frac{b-z}{b-a}(r'_1)^{-1} + \frac{z-a}{b-a}(r'_2)^{-1}\right) \qquad ((r'_j)^{-1} + r_j^{-1} = 1;\ j = 1, 2).$$

Let

$$H = \sum_{j=1}^{k} \lambda_j(\cdot, \phi_j)\phi_j$$

be the spectral decomposition of the operator H. Then

$$T_z UH^{\alpha+\beta z} = \sum_{j=1}^{k} \lambda_j^{\alpha+\beta z}(\cdot, \phi_j) T_z U\phi_j$$

and

$$f(z) = \sum_{j=1}^{k} \lambda_j^{\alpha+\beta z}(T_z U\phi_j, \phi_j) \qquad (a \leqq \mathrm{Re}\, z \leqq b).$$

From the representation obtained for the function $f(z)$ follows its holomorphness in the strip being considered. Moreover,

$$|f(a+iy)| \leqq |T_{a+iy}|_{r_1} |H^{\alpha+a\beta+i\beta y}|_{r'_1} \leqq C_1 |H^{\alpha+a\beta}|_{r'_1}.$$

Since for any nonnegative operator D we have

$$|D^l|_r = |D|_{lr}^l,$$

it follows that

$$|f(a+iy)| \leqq C_1 |H|_{r'_1(\alpha+a\beta)}^{\alpha+a\beta} = C_1 |K|_{r'}^{\alpha+a\beta} = C_1.$$

Estimating the values of $f(z)$ on the line $z = b + iy$ $(-\infty < y < \infty)$ in the same way, we obtain

$$|f(b+iy)| \leqq C_2 |H^{\alpha+b\beta}|_{r_2} = C_2 |K|_{r'}^{\alpha+b\beta} = C_2 \qquad (-\infty < y < \infty).$$

It is easily seen that the three lines theorem is applicable to the numerical function $f(z)$, by which we obtain

$$|f(x+iy)| \leqq C_1^{1-t_x} C_2^{t_x} \qquad (-\infty < y < \infty).$$

Since $\alpha + \beta x = 1$, for $y = 0$ we have

$$|f(x)| = |\mathrm{sp}(T_x K)| \leqq C_1^{1-t_x} C_2^{t_x}.$$

Since the last relation holds for any finite-dimensional operator K ($|K|_{r'} = 1$), we obtain, according to Lemma 12.1, $T_x \in \mathfrak{S}_r$ and

$$|T_x|_r \leqq C_1^{1-t_x} C_2^{t_x}.$$

The theorem is proved.

Theorem 13.1 was proved in collaboration with S. G. Kreĭn.

§14. The symmetrically-normed ideals $\mathfrak{S}_{\Pi}$ and $\mathfrak{S}_{\Pi}^{(0)}$

1. Let $\Pi = \{\pi_j\}_1^\infty$ be an arbitrary nonincreasing sequence of positive numbers, with $\pi_1 = 1$. We associate with the sequence Π the function

$$\Phi_{\Pi}(\xi) = \sup_n \left[\sum_{j=1}^n \xi_j^* \Big/ \sum_{j=1}^n \pi_j \right] \qquad (\xi = \{\xi_j\} \in \hat{c}).$$

It can be proved in an evident way that $\Phi_{\Pi}(\xi)$ is an s.n. function. From the definition of $\Phi_{\Pi}(\xi)$ it follows at once that its natural domain $c_{\Phi_{\Pi}} = c_{\Pi}$ consists of all vectors $\xi = \{\xi_j\} \in c_0$ for which

$$\sup_n \left[\sum_{j=1}^n \xi_j^* \Big/ \sum_{j=1}^n \pi_j \right] < \infty.$$

LEMMA 14.1. *If*

$$\sum_{j=1}^{\infty} \pi_j < \infty, \tag{14.1}$$

then the function $\Phi_{\Pi}(\xi)$ is equivalent to the maximal one. If

$$\lim_{n\to\infty} \pi_n > 0, \tag{14.2}$$

then the function $\Phi_{\Pi}(\xi)$ is equivalent to the minimal one. If

$$\sum_j \pi_j = \infty \quad \textit{and} \quad \lim_{n\to\infty} \pi_n = 0, \tag{14.3}$$

(*i.e. neither one of the conditions* (14.1) *and* (14.2) *is fulfilled*), *then the function $\Phi_{\Pi}(\xi)$ is not equivalent to either the minimal or the maximal one, and moreover in this case it is a binormalizing function.*

PROOF. In fact,

$$\Phi_{\Pi}(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) = n \Big/ \sum_{j=1}^{n} \pi_j,$$

since

$$\max_{1 \leq k < \infty} \left[\min(k, n) \Big/ \sum_{j=1}^{k} \pi_j \right] = n \Big/ \sum_{j=1}^{n} \pi_j.$$

It follows at once that if condition (14.1) is fulfilled, then

$$\sup_n \frac{n}{\Phi_{\Pi}(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots)} = \sum_{j=1}^{\infty} \pi_j < \infty,$$

and so the function $\Phi_{\Pi}(\xi)$ is equivalent to the maximal one.

If the condition (14.2) is fulfilled, then obviously

$$\sum_{j=1}^{n} \pi_j \geqq n\pi_{\infty},$$

where $\pi_{\infty} = \lim_{n \to \infty} \pi_n > 0$, and consequently

$$\sup_n \Phi_{\Pi}(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) \leqq 1/\pi_{\infty}.$$

Thus in this case the function $\Phi_{\Pi}(\xi)$ is equivalent to the minimal one.

If the second condition in (14.3) holds, then $\Pi = \{\pi_j\} \in \boldsymbol{c}_{\Pi}$; obviously

$$\Phi_{\Pi}(\pi_{m+1}, \pi_{m+2}, \cdots) = \sup_n \left[\sum_{j=1}^{n} \pi_{m+j} \Big/ \sum_{j=1}^{n} \pi_j \right] \leqq 1.$$

On the other hand,

$$\frac{\sum_{j=1}^{n} \pi_{m+j}}{\sum_{j=1}^{n} \pi_j} = \frac{\sum_{j=1}^{n+m} \pi_j}{\sum_{j=1}^{n} \pi_j} - \frac{\sum_{j=1}^{m} \pi_j}{\sum_{j=1}^{n} \pi_j} \geqq 1 - \frac{\sum_{j=1}^{m} \pi_j}{\sum_{j=1}^{n} \pi_j}. \tag{14.4}$$

If the first of the conditions (14.3) is fulfilled, then it follows from (14.4) that

$$\Phi_{\Pi}(\pi_{m+1}, \pi_{m+2}, \cdots) \geqq 1.$$

Thus when the first of the conditions (14.3) is fulfilled, we have

$$\Phi_{\Pi}(\pi_{m+1}, \pi_{m+2}, \cdots) = 1,$$

i.e. $\Phi_{\Pi}(\xi)$ is a binormalizing function. The lemma is proved.

The s.n. function $\Phi_{\Pi}(\xi)$ has the following characteristic property.

Let $\Pi = \{\pi_j\}_1^\infty$ be an arbitrary nonincreasing sequence of positive numbers with $\pi_1 = 1$. Then the function $\Phi_\Pi(\xi)$ is maximal among all s.n. functions $\Phi(\xi)$ having the property that for any $n = 1, 2, \cdots$

$$\Phi(\pi_1, \pi_2, \cdots, \pi_n, 0, 0, \cdots) \leqq 1. \tag{14.5}$$

In fact, for any vector $\xi = \{\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots\} \in \hat{c}$ we have

$$\sum_{j=1}^m \xi_j^* \leqq \Phi_\Pi(\xi) \sum_{j=1}^m \pi_j \qquad (m = 1, 2, \cdots),$$

and consequently, for any s.n. function satisfying the condition (14.5), we shall have, by virtue of its property V′),

$$\Phi(\xi) \leqq \Phi_\Pi(\xi)\,\Phi(\pi_1, \pi_2, \cdots, \pi_n, 0, 0, \cdots) \leqq \Phi_\Pi(\xi) \qquad (\xi \in \hat{c}).$$

2. From Lemma 14.1 it is clear that a new s.n. ideal $\mathfrak{S}_\Pi$ (different from $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$) is obtained only when the sequence $\{\pi_n\}_1^\infty$ $(\pi_1 = 1)$ satisfies the conditions (14.3). We shall call such a sequence *binormalizing*. The introduction of this terminology is justified by the following result.

THEOREM 14.1. *Let* $\Pi = \{\pi_j\}_1^\infty$ *be an arbitrary binormalizing sequence. Then the set* $\mathfrak{S}_\Pi$ *of all completely continuous operators* A *for which*

$$\sup_n \left[\sum_{j=1}^n s_j(A) \Big/ \sum_{j=1}^n \pi_j \right] < \infty$$

forms a nonseparable s.n. ideal with the norm

$$|A|_\Pi = |A|_{\Phi_\Pi} = \sup_n \left[\sum_{j=1}^n s_j(A) \Big/ \sum_{j=1}^n \pi_j \right]. \tag{14.6}$$

The s.n. ideal $\mathfrak{S}_\Pi$ *contains as a proper subset the separable s.n. ideal* $\mathfrak{S}_\Pi^{(0)}$, *consisting of all the operators* $A \in \mathfrak{S}_\Pi$ *for which*

$$\lim_{n\to\infty} \left[\sum_{j=1}^n s_j(A) \Big/ \sum_{j=1}^n \pi_j \right] = 0. \tag{14.7}$$

Also,

$$\inf_{X \in \mathfrak{S}_\Pi^{(0)}} |A - X|_\Pi = \varlimsup_{n\to\infty} \left[\sum_{j=1}^n s_j(A) \Big/ \sum_{j=1}^n \pi_j \right]. \tag{14.8}$$

PROOF. The first assertion of the theorem follows from Theorem 4.1, Corollary 6.1 and Lemma 14.1, if one takes into account that $\mathfrak{S}_\Pi = \mathfrak{S}_{\Phi_\Pi}$. To prove the remaining assertions, it is sufficient to show that

$$\inf_{X \in \mathfrak{K}} |A - X|_\Pi = \varlimsup_{n\to\infty} \left[\sum_{j=1}^n s_j(A) \Big/ \sum_{j=1}^n \pi_j \right]. \tag{14.8′}$$

In fact, it follows from this relation that $A \in \mathfrak{S}_{\Phi_{\text{II}}}^{(0)}$ if and only if (14.7) is fulfilled, whence $\mathfrak{S}_{\text{II}}^{(0)} = \mathfrak{S}_{\Phi_{\text{II}}}^{(0)}$. But then the relation (14.8) will be a consequence of the relation (14.8′).

To prove (14.8′), we consider the two functionals $l_1(A)$ and $l_2(A)$ $(A \in \mathfrak{S}_{\text{II}})$ defined by

$$l_1(A) = \lim_{m \to \infty} \sup_n \left[\sum_{j=1}^n s_{m+j}(A) \Big/ \sum_{j=1}^n \pi_j \right],$$

$$l_2(A) = \overline{\lim_{n \to \infty}} \left[\sum_{j=1}^n s_j(A) \Big/ \sum_{j=1}^n \pi_j \right].$$

According to Lemma 6.1,

$$l_1(A) = \inf_{K \in \mathfrak{R}} |A - K|_{\text{II}},$$

and consequently the relation (14.8′) will be proved as soon as it is shown that $l_1(A) = l_2(A)$ for any $A \in \mathfrak{S}_{\text{II}}$. This assertion, in turn, is a simple consequence of the fact that for any monotone decreasing sequence $\{s_j\}_1^\infty$ of nonnegative numbers tending to zero we have

$$(l_1 =) \lim_{m \to \infty} \sup_n \left[\sum_{j=1}^n s_{m+j} \Big/ \sum_{j=1}^n \pi_j \right] = \overline{\lim_{n \to \infty}} \left[\sum_{j=1}^n s_j \Big/ \sum_{j=1}^n \pi_j \right] (= l_2).$$

Indeed, since

$$\sum_{j=1}^n s_j \leqq \sum_{j=1}^m s_j + \sum_{j=1}^n s_{m+j} \qquad (n \geqq m;\ n, m = 1, 2, \cdots),$$

it follows that

$$\frac{\sum_{j=1}^n s_j}{\sum_{j=1}^n \pi_j} \leqq \frac{\sum_{j=1}^m s_j}{\sum_{j=1}^n \pi_j} + \sup_n \frac{\sum_{j=1}^n s_{m+j}}{\sum_{j=1}^n \pi_j}.$$

Thus

$$l_2 \leqq \sup_n \left[\sum_{j=1}^n s_{m+j} \Big/ \sum_{j=1}^n \pi_j \right].$$

Passing to the limit $m \to \infty$ we obtain

$$l_2 \leqq l_1. \tag{14.9}$$

On the other hand, for any $\epsilon > 0$ one can find n_0 such that

$$\sum_{j=1}^n s_j \Big/ \sum_{j=1}^n \pi_j < l_2 + \epsilon$$

whenever $n > n_0$. It follows that

$$\sum_{j=1}^{n} s_{m+j} \Big/ \sum_{j=1}^{n} \pi_j < l_2 + \epsilon \qquad (m = 1, 2, \cdots; \; n > n_0). \tag{14.10}$$

It is easily seen that for sufficiently large m the relation (14.10) will also hold for $n \leqq n_0$. Hence from (14.10) we get

$$l_1 \leqq l_2 + \epsilon. \tag{14.11}$$

Comparing the relations (14.9) and (14.11), we obtain $l_1 = l_2$. The theorem is proved.

REMARK 14.1. B. S. Mitjagin [1] has shown that for any binormalizing sequence $\Pi = \{\pi_j\}_1^\infty$ there exists an *intermediate* s.n. ideal $\tilde{\mathfrak{S}}_\Pi$, strictly included between $\mathfrak{S}_\Pi^{(0)}$ and $\mathfrak{S}_\Pi$, and that the norm in $\tilde{\mathfrak{S}}_\Pi$ is defined by the s.n. function $\Phi_\Pi(\xi)$.

3. We shall say that a binormalizing sequence $\Pi = \{\pi_j\}_1^\infty$ is *regular* if the following condition is fulfilled:

$$\sum_{j=1}^{n} \pi_j = O(n\pi_n) \qquad (n \to \infty). \tag{14.12}$$

We note that always

$$\sum_{j=1}^{n} \pi_j \geqq n\pi_n \qquad (n = 1, 2, \cdots).$$

It can be verified without difficulty that the sequence $\{n^{-\alpha}\}_1^\infty$ $(0 < \alpha < 1)$ is regular.

THEOREM 14.2. *If the sequence $\Pi = \{\pi_j\}_1^\infty$ is regular, then the s.n. ideal $\mathfrak{S}_\Pi$ coincides with the class of all operators $A \in \mathfrak{S}_\infty$ for which*

$$s_n(A) = O(\pi_n) \qquad (n \to \infty), \tag{14.13}$$

and the class $\mathfrak{S}_\Pi^{(0)}$ coincides with the class of all operators $A \in \mathfrak{S}_\infty$ for which

$$s_n(A) = o(\pi_n) \qquad (n \to \infty). \tag{14.14}$$

PROOF. In fact, every operator $A \in \mathfrak{S}_\infty$ for which (14.13) holds obviously belongs to the class $\mathfrak{S}_\Pi$. Conversely, if $A \in \mathfrak{S}_\Pi$, then

$$\sum_{j=1}^{n} s_j(A) \leqq |A|_\Pi \sum_{j=1}^{n} \pi_j \qquad (n = 1, 2, \cdots)$$

and consequently

$$s_n(A) \leqq |A|_\Pi \left\{ \frac{1}{n} \sum_{j=1}^{n} \pi_j \right\} = O(\pi_n) \qquad (n \to \infty).$$

If condition (14.14) is fulfilled, then obviously condition (14.7) is fulfilled, and thus $A \in \mathfrak{S}_{\Pi}^{(0)}$. Conversely, if condition (14.7) holds, then

$$\sum_{k=1}^{n} s_k(A) = o\left(\sum_{k=1}^{n} \pi_k\right) \qquad (n \to \infty),$$

or

$$s_n(A) = o\left(\frac{1}{n}\sum_{k=1}^{n} \pi_k\right) = o(\pi_n) \qquad (n \to \infty).$$

The theorem is proved.

4. Let us agree to call a function $L(\nu)$ $(a \leqq \nu < \infty;\ a > 0)$ *slowly varying* if it is positive and continuously differentiable, and if

$$\lim_{\nu \to \infty} \{\nu L'(\nu)/L(\nu)\} = 0. \tag{14.15}$$

As an example of a slowly varying function we can take any function

$$L(\nu) = c(\ln_1\nu)^{\alpha 1}(\ln_2\nu)^{\alpha 2} \cdots (\ln_k\nu)^{\alpha k},$$

where

$$\ln_1\nu = \ln\nu;\ \ln_r\nu = \ln(\ln_{r-1}\nu) \qquad (r = 2, 3, \cdots),$$

and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are arbitrary real numbers.

The following properties of such functions can be verified at once:

a) the sum and product of two slowly varying functions is also a slowly varying function;

b) if $L(\nu)$ is a slowly varying function, so is $[L(\nu)]^\alpha$ $(-\infty < \alpha < \infty)$;

c) for any $\epsilon > 0$, one can always find constants C'_ϵ, C''_ϵ for a given slowly varying function $L(\nu)$ such that

$$C'_\epsilon \nu^{-\epsilon} < L(\nu) < C''_\epsilon \nu^\epsilon \qquad (1 \leqq \nu < \infty).$$

Slowly varying functions are of interest in that they enable us to construct regular sequences Π. It turns out that

1. *A nonincreasing sequence* $\{\pi_j\}_1^\infty$ *is regular if it can be represented in the form*

$$\pi_n = n^{-1/p}L(n) \qquad (n = 1, 2, \cdots;\ 1 < p < \infty), \tag{14.16}$$

where $L(\nu)$ $(1 \leqq \nu < \infty;\ L(1) = 1)$ *is an arbitrary slowly varying function and the function* $\nu^{-1/p}L(\nu)$ *is nonincreasing.*

In fact, it follows at once from the property c) that the sequence (14.16) is binormalizing. Let us verify the regularity condition (14.12). By hypothesis the function $\nu^{-1/p}L(\nu)$ $(1 \leqq \nu < \infty)$ is nonincreasing; consequently

$$\sum_{k=1}^{n} k^{-1/p} L(k) \leqq \int_1^n t^{-1/p} L(t)\, dt + 1 \quad (n = 1, 2, \cdots). \tag{14.17}$$

Integrating the right side of this inequality by parts, we obtain

$$\int_1^n t^{-1/p} L(t)\, dt \leqq \frac{1}{1 - 1/p} \left(n^{1-1/p} L(n) + \int_1^n t^{1-1/p} |L'(t)|\, dt\right). \tag{14.18}$$

Let the number $m \geqq 1$ be chosen so that for all $t \geqq m$

$$t|L'(t)|/L(t) < (1 - 1/p)/2.$$

Then for $n > m$ we have

$$\begin{aligned} \int_1^n t^{1-1/p} |L'(t)|\, dt &= \int_1^n t^{-1/p} L(t) \frac{t|L'(t)|}{L(t)}\, dt \\ &\leqq \int_1^m t^{1-1/p} |L'(t)|\, dt + \frac{1 - 1/p}{2} \int_1^n t^{-1/p} L(t)\, dt. \end{aligned} \tag{14.19}$$

Comparing (14.18) and (14.19), we obtain

$$\int_1^n t^{-1/p} L(t)\, dt \leqq \frac{2}{1 - 1/p} \left(n^{1-1/p} L(n) + \int_1^m t^{1-1/p} |L'(t)|\, dt\right). \tag{14.20}$$

Since $n^{1-1/p} L(n) \to \infty$ as $n \to \infty$, from (14.20) and (14.17) follows

$$\sum_{k=1}^{n} k^{-1/p} L(k) = O(n^{1-1/p} L(n)),$$

which shows the regularity of the sequence (14.16).

§15. The symmetrically-normed ideals $\mathfrak{S}_\pi$ and their connection with $\mathfrak{S}_{\mathrm{II}}$ and $\mathfrak{S}_{\mathrm{II}}^{(0)}$

1. With every nonincreasing sequence $\Pi = \{\pi_j\}_1^\infty$ $(\pi_1 = 1)$ of positive numbers we also associate the function

$$\Phi_\pi(\xi) = \sum_j \pi_j \xi_j^* \qquad (\xi = \{\xi_j\} \in \hat{c}). \tag{15.1}$$

The function $\Phi_\pi(\xi)$ is an s.n. function. In fact, it is obvious that $\Phi_\pi(\xi)$ has the properties I′)—IV′) of an s.n. function.

From the identity

$$\sum_{j=1}^{n} \pi_j \xi_j = \pi_n \eta_n + \sum_{j=1}^{n-1} (\pi_j - \pi_{j+1}) \eta_j \qquad (\xi \in \hat{k}),$$

where

$$\eta_j = \sum_{r=1}^{j} \xi_r \qquad (j = 1, 2, \cdots, n),$$

it follows that $\Phi_\pi(\xi)$ has the property V′).

The natural domain $c_\pi = c_{\Phi_\pi}$ of the function $\Phi_\pi(\xi)$ consists, obviously, of all vectors $\xi = \{\xi_j\} \in c_0$ for which

$$\sum_{j=1}^{\infty} \pi_j \xi_j^* < \infty.$$

LEMMA 15.1. *If*

$$\sum_{j=1}^{\infty} \pi_j < \infty, \tag{15.2}$$

then the function $\Phi_\pi(\xi)$ *is equivalent to the minimal one. If*

$$\lim_{n\to\infty} \pi_n > 0, \tag{15.3}$$

then the function $\Phi_\pi(\xi)$ *is equivalent to the maximal one. If* $\Pi = \{\pi_j\}_1^\infty$ *is a binormalizing sequence, i.e.*

$$\sum_{j=1}^{\infty} \pi_j = \infty \quad \textit{and} \quad \lim_{n\to\infty} \pi_n = 0, \tag{15.4}$$

then the function $\Phi_\pi(\xi)$ *is not equivalent either to the minimal or the maximal one; the function* $\Phi_\pi(\xi)$ *is always mononormalizing.*

PROOF. In fact,

$$\Phi_\pi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) = \sum_{j=1}^{n} \pi_j, \tag{15.5}$$

and consequently when the condition (15.2) is fulfilled the function $\Phi_\pi(\xi)$ is equivalent to the minimal one. Since

$$\lim_{n\to\infty} \frac{n}{\Phi_\pi(\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots)} = \frac{1}{\lim\limits_{n\to\infty} \left[\dfrac{1}{n} \sum\limits_{j=1}^{n} \pi_j\right]} = \frac{1}{\lim\limits_{n\to\infty} \pi_n}, \tag{15.6}$$

we see that when the condition (15.3) is fulfilled, the function $\Phi_\pi(\xi)$ is equivalent to the maximal one.

Finally, suppose the condition (15.4) to be fulfilled; then it follows from (15.5) and (15.6) that the function $\Phi_\pi(\xi)$ is not equivalent either to the minimal or the maximal one.

Let ξ be an arbitrary vector from c_π; then for any positive ϵ one can find m such that

$$\sum_{j=m+1}^{\infty} \pi_j \xi_j^* < \epsilon/2.$$

Since $\lim \xi_j^* = 0$, we can select an n such that

$$\sum_{j=1}^{m} \pi_j \xi_{j+n}^* < \epsilon/2.$$

Taking into account that

$$\sum_{j=1}^{\infty} \pi_j \xi_{j+n}^* \leqq \sum_{j=1}^{m} \pi_j \xi_{n+j}^* + \sum_{j=m+1}^{\infty} \pi_j \xi_j^*,$$

we obtain

$$\sum_{j=1}^{\infty} \pi_j \xi_{n+j}^* < \epsilon.$$

Hence

$$\lim_{n\to\infty} \Phi_\pi(\xi_{n+1}^*, \xi_{n+2}^*, \cdots) = 0 \quad (\xi \in \boldsymbol{c}_\pi),$$

and so the function $\Phi_\pi(\xi)$ is mononormalizing. The lemma is proved.

2. As a corollary of general theorems and of the lemma just proved, we obtain the following theorem.

THEOREM 15.1. *Let $\Pi = \{\pi_j\}_1^\infty$ be an arbitrary binormalizing sequence. Then the set $\mathfrak{S}_\pi$ of all completely continuous operators A for which*

$$\sum_{j=1}^{\infty} \pi_j s_j(A) < \infty$$

forms a separable s.n. ideal with the norm

$$|A|_\pi = \sum_{j=1}^{\infty} \pi_j s_j(A). \tag{15.7}$$

The ideal of finite-dimensional operators is dense in $\mathfrak{S}_\pi$, and

$$\min_{A\in\mathfrak{K}_n} |A - K|_\pi = \sum_{j=1}^{\infty} \pi_j s_{n+j}(A) \quad (A \in \mathfrak{S}_\pi).$$

3. It will be proved below that the class $\mathfrak{S}_\pi$ is the space adjoint to $\mathfrak{S}_\Pi^{(0)}$. We precede this result by a lemma.

LEMMA 15.2. *Let $\mathfrak{k}_n$ be the cone of real n-dimensional coordinate space, consisting of all vectors $\xi = \{\xi_j\}_1^n$ for which $\xi_1 \geqq \xi_2 \geqq \cdots \geqq \xi_n \geqq 0$, and let $\{\pi_j\}_1^n$ be an arbitrary system of n positive numbers. Then*

$$\sum_{j=1}^{n} \eta_j \xi_j \leqq \sum_{j=1}^{n} \pi_j \xi_j \max_r \left[\sum_{j=1}^{r} \eta_j \Big/ \sum_{j=1}^{r} \pi_j \right] \tag{15.8}$$

for any pair of vectors $\xi = \{\xi_j\}_1^n$, $\eta = \{\eta_j\}_1^n$ from $\mathfrak{k}_n$. For every nonzero vector $\xi \in \mathfrak{k}_n$ $(\eta \in \mathfrak{k}_n)$ there exists a nonzero vector $\eta \in \mathfrak{k}_n$ $(\xi \in \mathfrak{k}_n)$ for which equality holds in (15.8).

PROOF. Let the vector η be fixed. We denote by Λ the convex set which is the intersection of the cone $\mathfrak{k}_n$ and the hyperplane $\sum_{j=1}^{n} \pi_j \xi_j = 1$. It is obvious that the proof of the lemma reduces to the proof of the equality

$$\max_{\xi \in \Lambda} f(\xi) = \max_{r} \left[\sum_{j=1}^{r} \eta_j \Big/ \sum_{j=1}^{r} \pi_j \right], \tag{15.9}$$

where $f(\xi) = \sum_{j=1}^{n} \xi_j \eta_j$. As is known, a linear functional always assumes its maximum value on Λ at extreme points of Λ (points which are not interior points of any interval belonging to Λ).

To determine all the extreme points of Λ we associate with every vector $\xi = \{\xi_j\} \in \Lambda$ a vector $x = \{x_j\}$ according to the rule $x_j = \xi_j - \xi_{j+1}$ $(j = 1, 2, \cdots, n-1)$, $x_n = \xi_n$. For this correspondence, as ξ ranges over Λ, the vector x ranges over a polyhedron Λ', consisting of all vectors $x = \{x_j\}_1^n$ $(x_j \geqq 0)$ for which

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{j} \pi_k \right) x_j = 1. \tag{15.9'}$$

Since the indicated transformation $\xi \to x$ $(\xi \in \Lambda,\ x \in \Lambda')$ is one-to-one and linear, it establishes a one-to-one correspondence between the extreme points of Λ and Λ'. The body Λ' is the simplex along which the hyperplane (15.9′) intersects the hyperoctant of vectors with non-negative coordinates; consequently the extreme points of Λ' will be the vertices of the simplex, i.e. the vectors only one of whose coordinates is different from zero.

It follows that the extreme points of Λ are all the vectors of the form

$$\xi = \{\underbrace{a, a, \cdots, a}_{r}, 0, 0, \cdots, 0\},$$

where

$$a = \left(\sum_{j=1}^{r} \pi_j \right)^{-1}.$$

For such a point

$$f(\xi) = \sum_{j=1}^{r} \eta_j \Big/ \sum_{j=1}^{r} \pi_j.$$

The lemma is proved.[20]

4. According to the result of §11.2 it follows from the lemma that for any nonincreasing sequence $\Pi = \{\pi_j\}_1^\infty$ $(\pi_1 = 1)$ of nonnegative numbers the functions $\Phi_\Pi(\xi)$ and $\Phi_\pi(\xi)$ are mutually adjoint, i.e.,

[20] The authors are obliged to F. V. Širokov for a simplification of the proof of this lemma.

$$\Phi_\Pi^*(\xi) = \Phi_\pi(\xi) \quad \text{and} \quad \Phi_\pi^*(\xi) = \Phi_\Pi(\xi).$$

Thus, by Theorem 12.2, the following theorem is valid.

THEOREM 15.2. *Let* $\Pi = \{\pi_j\}_1^\infty$ *be a binormalizing sequence. Then, for the triple of spaces* $\mathfrak{S}_\Pi^{(0)}$, $\mathfrak{S}_\pi$ *and* $\mathfrak{S}_\Pi$, *each space is the adjoint of the preceding one.*

REMARK 15.1. The basic theorems 14.1, 15.1 and 15.2 are formulated under the assumption that $\Pi = \{\pi_j\}_1^\infty$ is a binormalizing sequence. We have put aside the case of "little interest" for which

1) $$\lim_{j\to\infty} \pi_j > 0$$

or

2) $$\sum_{j=1}^\infty \pi_j < \infty.$$

In the first case the s.n. ideal $\mathfrak{S}_\pi(=\mathfrak{S}_{\Phi_\pi})$ coincides elementwise with $\mathfrak{S}_1$; $\mathfrak{S}_\Pi^{(0)} = \mathfrak{S}_{\Phi_\Pi}^{(0)}$ coincides elementwise with $\mathfrak{S}_\infty$, while $\mathfrak{S}_\pi$ is the adjoint of $\mathfrak{S}_\Pi^{(0)}$, and the adjoint of $\mathfrak{S}_\pi$ will be the ring $\mathfrak{R}_{\Phi_\Pi}$ (cf. §12), which coincides elementwise with the entire ring $\mathfrak{R}$. The reader can easily analyze, on the basis of Lemmas 14.1 and 15.1, the situation which holds in the second case.

5. In a number of questions of the theory of nonselfadjoint operators (cf. Macaev [2], Gohberg and Kreĭn [3, 4], Brodskiĭ, Gohberg, Kreĭn and Macaev [1]), in the theory of the abstract triangular integral, in questions on the completeness of the root vectors of an operator, etc., a special role is played by the s.n. ideal $\mathfrak{S}_\omega$, where $\Omega = \{1/(2n-1)\}_1^\infty$, and the s.n. ideal $\mathfrak{S}_\Omega$ which is adjoint to $\mathfrak{S}_\omega$. The s.n. ideal $\mathfrak{S}_\omega$ was introduced by V. I. Macaev [2].

The introduction of this space served the authors as an incentive for the study of the general spaces $\mathfrak{S}_\Pi^{(0)}$, $\mathfrak{S}_\pi$ and $\mathfrak{S}_\Pi$ and the establishing of connections between them. It is easily seen that the space $\mathfrak{S}_\omega$ contains all the spaces $\mathfrak{S}_p$ $(1 \leqq p < \infty)$ and that

$$|A|_\omega \leqq \left(\sum_{j=1}^\infty \left(\frac{1}{2j-1}\right)^q\right)^{1/q} |A|_p \qquad (A \in \mathfrak{S}_p;\ p^{-1}+q^{-1}=1),$$

and that the nonseparable space $\mathfrak{S}_\Omega$ lies in the gap between the spaces $\mathfrak{S}_1$ and $\mathfrak{S}_p$ $(1<p\leqq\infty)$, i.e. it contains $\mathfrak{S}_1$ and is contained in all the spaces $\mathfrak{S}_p$ $(1<p\leqq\infty)$, and

$$|A|_p \leqq \left(\sum_{j=1}^\infty \left(\frac{1}{2j-1}\right)^p\right)^{1/p} |A|_\Omega \qquad (A \in \mathfrak{S}_\Omega).$$

6. Returning to the s.n. ideals $\mathfrak{S}_\pi$ and $\mathfrak{S}_\Pi$, we mention two more

simple properties of $\mathfrak{S}_\pi$ and $\mathfrak{S}_\Pi$, whose proofs are left to the reader.

1. *In order that, for two nonincreasing sequences* $\Pi' = \{\pi_j'\}$, $\Pi'' = \{\pi_j''\}$, *the normed ideals* $\mathfrak{S}_{\pi'}$ *and* $\mathfrak{S}_{\pi''}$, *as well as* $\mathfrak{S}_{\Pi'}$ *and* $\mathfrak{S}_{\Pi''}$, *shall coincide elementwise, it is necessary and sufficient that*

$$\inf_n \left[\sum_{j=1}^n \pi_j' \Big/ \sum_{j=1}^n \pi_j'' \right] > 0 \quad \text{and} \quad \sup_n \left[\sum_{j=1}^n \pi_j' \Big/ \sum_{j=1}^n \pi_j'' \right] < \infty .$$

2. *If*

$$\sum_{j=1}^\infty \pi_j^p < \infty \qquad (p > 1),$$

then the class $\mathfrak{S}_p$ *contains the class* $\mathfrak{S}_\Pi$, *and* $\mathfrak{S}_q$ $(p^{-1} + q^{-1} = 1)$ *is contained in the class* $\mathfrak{S}_\pi$; *moreover,*

$$|A|_p \leqq \left(\sum_{j=1}^\infty \pi_j^p \right)^{1/p} |A|_\Pi \qquad (A \in \mathfrak{S}_\Pi) \tag{15.10}$$

and

$$|A|_\pi \leqq \left(\sum_{j=1}^\infty \pi_j^p \right)^{1/p} |A|_q \qquad (A \in \mathfrak{S}_q). \tag{15.11}$$

§16. Another interpolation theorem

Let, as before, $\Pi = \{\pi_j\}_1^\infty$ be an arbitrary binormalizing sequence. Along with it, the sequence $\Pi_p = \{\pi_j^{1/p}\}_1^\infty$ $(1 \leqq p < \infty)$ will also be binormalizing. Let us denote by $\mathfrak{S}_{\Pi;p}$ $(1 \leqq p < \infty)$ the s.n. ideal $\mathfrak{S}_{\Pi_p}$, and by $|A|_{\Pi;p}$ the norm in this space. Similarly, we denote by $\mathfrak{S}_{\pi;p}$ the s.n. ideal of all operators $A \in \mathfrak{S}_\infty$ for which

$$|A|_{\pi;p} = \sum_{n=1}^\infty \pi_n^{1/p} s_n(A) < \infty ,$$

with the norm $|A|_{\pi;p}$.

THEOREM 16.1. *Let* T_z $(\in \mathfrak{S}_\infty)$ *be an operator-function holomorphic in the strip* $a_1 \leqq \operatorname{Re} z \leqq a_2$. *If for some* p_1, p_2 $(1 \leqq p_2 < p_1 < \infty)$

$$|T_{a_j + iy}|_{\Pi;p_j} \leqq C_j \qquad (-\infty < y < \infty;\ j = 1,2),$$

and if for any pair of vectors $f, g \in \mathfrak{H}$

$$\ln |(T_z f, g)| \leqq N_{f,g} \exp(k_{f,g} |\operatorname{Im} z|) \qquad (a_1 < \operatorname{Re} z < a_2),$$

where $0 \leqq k_{f,g} < \pi/(a_2 - a_1)$, *then for any intermediate* x $(a_1 < x < a_2)$

$$|T_{x+iy}|_{\Pi;p_x} \leqq C_1^{1-t_x} C_2^{t_x} \qquad (-\infty < y < \infty), \tag{16.1}$$

where

$$p_x^{-1} = (1 - t_x)p_1^{-1} + t_x p_2^{-1}, \quad t_x = (x - a_1)/(a_2 - a_1).$$

PROOF. Just as in the proof of Theorem 13.1 we can establish that this theorem will be proved as soon as the relation (16.1) is proved for the special case $y = 0$.

Starting with an arbitrary finite-dimensional operator K whose Schmidt expansion has the form

$$K = \sum_{j=1}^{m} s_j(K)(\cdot, \psi_j)\phi_j,$$

we form the analytic operator-function

$$K_z = \sum_{j=1}^{m} \pi_j^z s_j(K)(\cdot, \psi_j)\phi_j. \tag{16.2}$$

It is obvious that for any p, r $(1 \leqq p, r < \infty;\ 1/p + 1/r \leqq 1)$

$$|K_{p^{-1}+iy}|_{\pi;r} = |K|_{\pi;l};\quad l^{-1} = p^{-1} + r^{-1}. \tag{16.3}$$

Let us consider the function

$$f(z) = \mathrm{sp}(T_z K_{v(z)}) \qquad (a_1 \leqq \mathrm{Re}\, z \leqq a_2),$$

where

$$v(z) = p_x^{-1} - \left(\frac{z - a_1}{a_2 - a_1} p_2^{-1} + \frac{a_2 - z}{a_2 - a_1} p_1^{-1}\right).$$

It follows from (16.2) that

$$f(z) = \sum_{j=1}^{m} \pi_j^{v(z)} s_j(K)(T_z\phi_j, \psi_j) \qquad (a_1 \leqq \mathrm{Re}\, z \leqq a_z).$$

Thus the function $f(z)$ is holomorphic in the strip which is being considered. Moreover,

$$|f(a_1 + iy)| \leqq |T_{a_1+iy}|_{\Pi;p_1} |K_{v(a_1+iy)}|_{\pi;p_1} \leqq C_1 |K_{v(a_1+iy)}|_{\pi;p_1}.$$

Since $\mathrm{Re}\, v(a_1 + iy) = p_x^{-1} - p_1^{-1}$, it follows from (16.3) that

$$|K_{v(a_1+iy)}|_{\pi;p_1} = |K|_{\pi;p_x} \qquad (-\infty < y < \infty),$$

and consequently

$$|f(a_1 + iy)| \leqq C_1 |K|_{\pi;p_x} \qquad (-\infty < y < \infty).$$

Estimating in the same way the values of the function $f(z)$ on the line $z = a_2 + iy$ $(-\infty < y < \infty)$, we obtain

$$|f(a_2+iy)| \leqq C_2|K|_{\pi;p_x} \qquad (-\infty<y<\infty).$$

It is easily seen that the three lines theorem is applicable to the function $f(z)$, from which

$$|f(x+iy)| \leqq C_1^{1-t_x}C_2^{t_x}|K|_{\pi;p_x} \qquad (-\infty<y<\infty).$$

Setting $y=0$ and taking into account that $v(x)=0$, we obtain

$$|f(x)| = |\operatorname{sp}(T_xK)| \leqq C_1^{1-t_x}C_2^{t_x}|K|_{\pi;p_x}.$$

Bearing in mind, finally, that the set of all finite-dimensional operators $\mathfrak{K}$ is dense in $\mathfrak{S}_{\pi;p_x}$ and that the space $\mathfrak{S}_{\Pi;p_x}$ is the adjoint of the space $\mathfrak{S}_{\pi;p_x}$, we obtain

$$|T_x|_{\Pi;p_x} \leqq C_1^{1-t_x}C_2^{t_x}.$$

The theorem is proved.

§ 17. Conical norms in the real Banach spaces of operators $\hat{\mathfrak{S}}$.

1. Let $\mathfrak{S}$ be an arbitrary s.n. ideal. In this section we shall denote the norm $|A|_{\mathfrak{S}}$ of an operator $A\in\mathfrak{S}$ by $\|A\|$.

We denote by $\hat{\mathfrak{S}}$ the set of all selfadjoint operators from $\mathfrak{S}$. It is obvious that $\hat{\mathfrak{S}}$ forms a real Banach space with the norm $\|A\|$. It is easily verified that if $\mathfrak{S}$ is a separable s.n. ideal and $\mathfrak{S}^*$ is the space adjoint to $\mathfrak{S}$, then the space $\hat{\mathfrak{S}}^*$ of all selfadjoint operators from $\mathfrak{S}^*$ is adjoint to $\hat{\mathfrak{S}}$. In particular, if the space $\mathfrak{S}$ is reflexive, so is the space $\hat{\mathfrak{S}}$.

To obtain, further on, certain precise estimates, we need new norms in the spaces $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{S}}^*$. The introduction of these norms is connected with certain ideas of the geometry of cones in Banach spaces (see Kreĭn and Rutman [1]).

2. A set K of vectors of a Banach space $\mathfrak{B}$ is called a *cone*, if: 1) it is closed, 2) for any pair of vectors $x,y\in K$ and any pair of nonnegative numbers α, β the vector $\alpha x+\beta y$ belongs to K, and 3) from $x\in K$ and $x\neq 0$ follows $-x\notin K$.

A cone is said to be *pointed* if for any pair of vectors $x,y\in K$,

$$|x| \leqq |x+y|.$$

The last condition expresses, as it were, that the angle between any two vectors $x,y\in K$ does not exceed $\pi/2$. A cone K is said to be *reproducing*, if any vector $x\in\mathfrak{B}$ can be represented in the form $z=x-y$, where $x,y\in K$.

Let K be a reproducing cone; then the set of all continuous linear functionals $f\in\mathfrak{B}^*$ having the property that $f(x)\geqq 0$ for all $x\in K$ also

forms a cone (in the space $\mathfrak{B}^*$). This cone is said to be adjoint to the cone K and is denoted by K^*. Let K be a reproducing cone of the space $\mathfrak{B}$. The formulas

$$|z|_{K;p} = \inf_{x,y\in K;\, x-y=z} (|x|^p + |y|^p)^{1/p} \quad (1 \leqq p \leqq \infty) \tag{17.1}$$

define "conical" norms in the space $\mathfrak{B}$ which are topologically equivalent to the original norm; for $p = \infty$ (17.1) is to be understood as

$$|z|_{K;\infty} = \inf_{x,y\in K;\, x-y=z} \max(|x|, |y|).$$

If the adjoint cone K^* corresponding to the reproducing cone K is itself reproducing, then to the norm $|z|_{K;p}$ $(1 \leqq p \leqq \infty)$ there will correspond, in the adjoint space, the conical norm $|f|_{K^*;q}$ $(p^{-1} + q^{-1} = 1; f \in \mathfrak{B}^*)$.

In the following we shall not make use of these propositions from the theory of cones; rather we shall illustrate them, with proofs, on the example of cones in the space $\mathfrak{S}$.

3. Let us denote by $\mathscr{K}$ the set of all nonnegative operators from $\hat{\mathfrak{S}}$. We shall show that

1. *The set $\mathscr{K}$ is a pointed reproducing cone.*

In fact, that $\mathscr{K}$ is a cone is obvious. If $X, Y \in \mathscr{K}$, then by Lemma II.1.1

$$s_j(X+Y) = \lambda_j(X+Y) \geqq \lambda_j(X) = s_j(X) \qquad (j = 1, 2, \cdots),$$

whence

$$\|X\| \leqq \|X+Y\|,$$

i.e. $\mathscr{K}$ is a pointed cone.

Let $Z \in \hat{\mathfrak{S}}$ and let

$$Z = \sum_{j=1}^{\infty} \lambda_j(Z)(\cdot, e_j)e_j$$

be its spectral decomposition. Obviously,

$$Z = Z_+ - Z_-, \tag{17.2}$$

where

$$Z_+ = \sum_{\lambda_j>0} \lambda_j(Z)(\cdot, e_j)e_j; \quad Z_- = -\sum_{\lambda_j<0} \lambda_j(Z)(\cdot, e_j)e_j.$$

Since

$$\lambda_j(Z_+) \leqq s_j(Z) \text{ and } \lambda_j(Z_-) \leqq s_j(Z) \qquad (j = 1, 2, \cdots),$$

we have $Z_\pm \in \hat{\mathfrak{S}}$, and moreover

$$\max(\|Z_+\|, \|Z_-\|) \leqq \|Z\|. \tag{17.3}$$

Thus $Z_\pm \in \mathscr{K}$, and the assertion is proved.

The decomposition (17.2) has the following important property.

2. *Let*

$$Z = X - Y \qquad (Z \in \mathfrak{S};\ X, Y \in \mathscr{K});$$

then

$$\|Z_+\| \leqq \|X\| \quad \textit{and} \quad \|Z_-\| \leqq \|Y\|.$$

This result follows at once from Lemma II.1.2.

3. *The cone* $\mathscr{K} = \mathscr{K}_{\mathfrak{S}}$ *has as its adjoint the cone* $\mathscr{K}^* = \mathscr{K}_{\mathfrak{S}^*}$.

This result follows from Theorem 8.3 and Remark 8.1.

In our case the equality (17.1), which defines the conical norm $\|Z\|_{\mathscr{K};p}$ $(1 \leqq p \leqq \infty)$, assumes the form

$$\|Z\|_{\mathscr{K};p} = (\|Z_+\|^p + \|Z_-\|^p)^{1/p} \qquad (1 \leqq p \leqq \infty;\ Z \in \mathfrak{S}). \tag{17.1$'$}$$

Starting from (17.1′), it is easy to deduce that the functional $\|Z\|_{\mathscr{K};p}$ $(1 \leqq p \leqq \infty)$ has all the properties of a norm in $\mathfrak{S}$.

From the relation

$$\|Z\| \leqq \|Z_+\| + \|Z_-\| \leqq \sqrt[q]{2}\,\|Z\|_{\mathscr{K};p} \qquad (1 \leqq p \leqq \infty; p^{-1} + q^{-1} = 1)$$

and the relation

$$\frac{1}{\sqrt[p]{2}}\,\|Z\|_{\mathscr{K};p} \leqq \max(\|Z_+\|, \|Z_-\|) \leqq \|Z\| \quad (1 \leqq p \leqq \infty),$$

which follows from (17.3), it follows that all the conical norms $\|Z\|_{\mathscr{K};p}$ $(1 \leqq p \leqq \infty)$ are topologically equivalent to the original norm.

THEOREM 17.1. *Every one of the conical norms* $\|Z\|_{\mathscr{K};p}$ $(1 \leqq p \leqq \infty)$ *is topologically equivalent to the original norm* $\|Z\|$. *To the norm* $\|Z\|_{\mathscr{K};p}$ $(1 \leqq p \leqq \infty)$ *there corresponds the norm* $\|Z\|_{\mathscr{K}^*;q}$ $(p^{-1} + q^{-1} = 1)$ *in the adjoint space* $\mathfrak{S}^*$.

PROOF. The first assertion of the theorem has already been proved. We shall prove the second. Let $F(Z)$ be an arbitrary continuous linear functional on $\mathfrak{S}$; then

$$F(Z) = \mathrm{sp}(AZ) \qquad (Z \in \mathfrak{S}),$$

where $A \in \mathfrak{S}^*$. Since

$$\mathrm{sp}(AZ) = \mathrm{sp}(A_+Z_+) + \mathrm{sp}(A_-Z_-) - [\mathrm{sp}(A_+Z_-) + \mathrm{sp}(A_-Z_+)]$$

and

$$\mathrm{sp}(A_+Z_\pm) + \mathrm{sp}(A_-Z_\mp) \leqq \|A\|_{\mathscr{K}^*;q}\,\|Z\|_{\mathscr{K};p},$$

where $p^{-1}+q^{-1}=1$, it follows that

$$|\mathrm{sp}(AZ)| \leqq \|A\|_{\mathscr{K}^*;q}\|Z\|_{\mathscr{K};p} \quad (Z\in\hat{\mathfrak{S}}).$$

If $Z\in\hat{\mathfrak{S}}$ satisfies the condition $\mathfrak{R}(Z_\pm)=\mathfrak{R}(A_\pm)$, then $A_+Z_- = A_-Z_+ = 0$ and hence

$$\mathrm{sp}(AZ)=\mathrm{sp}(A_+Z_+)+\mathrm{sp}(A_-Z_-).$$

Following this it is not hard to show that for any $A\in\hat{\mathfrak{S}}^*$

$$\sup_Z \frac{|\mathrm{sp}(AZ)|}{\|Z\|_{\mathscr{K};p}} = \|A\|_{\mathscr{K}^*;q}.$$

The theorem is proved.

For the case of the spaces $\mathfrak{S}_{\Pi}^{(0)}$, $\mathfrak{S}_\pi$, $\mathfrak{S}_\Pi$ a special role is played (see the report by Brodskiĭ, Gohberg, Kreĭn and Macaev [1]) by the mutually adjoint conical norms

$$|Z|_{\pi;\mathscr{K}} = \sum_{j=1}^{\infty}\pi_j(\lambda_j^+(Z)+\lambda_j^-(Z)) \qquad (Z\in\hat{\mathfrak{S}}_\pi)$$

and

$$|Z|_{\Pi;\mathscr{K}} = \max\left\{\sup_n \frac{\sum_{j=1}^n \lambda_j^+(Z)}{\sum_{j=1}^n \pi_j},\ \sup_n \frac{\sum_{j=1}^n \lambda_j^-(Z)}{\sum_{j=1}^n \pi_j}\right\} \quad (Z\in\hat{\mathfrak{S}}_\Pi),$$

where

$$\lambda_j^+(Z)=\lambda_j(Z_+),\ \lambda_j^-(Z)=\lambda_j(Z_-) \qquad (j=1,2,\cdots).$$

CHAPTER IV

INFINITE DETERMINANTS AND RELATED ANALYTIC METHODS

We have already been led to call upon methods of the theory of analytic functions in previous chapters. In this chapter we shall make use of deeper and more varied methods of the theory of functions, for the most part related to the theory of growth of entire functions. Some of these have not yet had time to find a place in books on the theory of functions, and are discussed only in journals.

It is noteworthy that a number of these results arose in connection with the demands of the theory of operators.

The channels for the use in this chapter of theorems of the theory of functions are infinite determinants and perturbation determinants, general results on which are presented in the first three sections.

Sections 3—10 discuss results of M. G. Kreĭn, essentially announced in the notes [**8**, **9**], and §11 discusses results of V. I. Macaev [**3**].

The statements of the theorems of the theory of functions which are used are presented in the appropriate places; as concerns their proofs, we have been obliged to restrict ourselves to the appropriate references.

We remark that the theory of perturbation determinants finds important application in the perturbation theory of selfadjoint (bounded and unbounded) operators (cf. M. G. Kreĭn [**6**, **12**]). However these results fall outside the scope of the book.

A number of relations of the theory of perturbation determinants can be traced as far back as an old paper by H. Bateman [**1**] on integral operators.

§1. The characteristic determinant of a nuclear operator[1)]

1. Let $K = \sum_{j=1}^{n} (\cdot, \psi_j)\, \phi_j$ be an arbitrary finite-dimensional operator of dimension $\leqq n$. We denote by $\mathfrak{L}$ an arbitrary finite-dimensional subspace which contains the range of the operators K and K^*. Obviously $\mathfrak{L}$ is an invariant subspace of K, and K vanishes on the orthogonal complement of $\mathfrak{L}$. Let $\{\chi_j\}_1^m$ be an orthonormal basis of the subspace $\mathfrak{L}$; we denote by $\det(I-K)$ the determinant of the matrix $\| \delta_{jk} - (K\chi_j, \chi_k) \|_1^m$. As is known, this de-

1) All the basic results of the first two sections are known. Our presentation of them has some points of contact with a number of papers: Hille and Tamarkin [**1**], Lidskiĭ [**5**], Kuroda [**2**] and others.

terminant does not depend upon the choice of the subspace $\mathfrak{L}$ and the basis in it, since always

$$\det(I-K)=\prod_{j=1}^{\nu(K)}(1-\lambda_j(K)).$$

This equality suggests that the determinant $\det(I-A)$ of any operator $A\in\mathfrak{S}_1$ should be defined by the formula

$$\det(I-A)=\prod_{j=1}^{\nu(A)}(1-\lambda_j(A)). \tag{1.1}$$

The product on the right side of (1.1) converges, since, for any $A\in\mathfrak{S}_1$,

$$\sum_{j=1}^{\nu(A)}|\lambda_j(A)|\leqq|A|_1.$$

If A is a Volterra operator, we put $\det(I-A)=1$.

The determinant

$$\det(I-\mu A)=\prod_{j=1}^{\nu(A)}(1-\mu\lambda_j(A))\qquad(A\in\mathfrak{S}_1)$$

will be called the *characteristic determinant* of the operator A, and will be denoted by $D_A(\mu)$.

1. *The characteristic determinant $D_A(\mu)$ of any operator $A\in\mathfrak{S}_1$ is an entire function of genus zero,*[2] *and*

$$|D_A(\mu)|\leqq\prod_{j=1}^{\infty}(1+|\mu|\,s_j(A))\leqq e^{|\mu||A|_1}. \tag{1.2}$$

It remains to clarify (1.2). We have

$$|D_A(\mu)|\leqq\prod_{j=1}^{\nu(A)}(1+|\mu|\,|\lambda_j(A)|);$$

on the other hand, by Corollary II.3.1

$$\prod_{j=1}^{\nu(A)}(1+|\mu|\,|\lambda_j(A)|)\leqq\prod_{j=1}^{\infty}(1+|\mu|\,s_j(A)).$$

By virtue of the familiar inequality $1+x\leqq e^x$,

$$\prod_{j=1}^{\infty}(1+|\mu|\,s_j(A))\leqq\exp\left[|\mu|\sum_{j=1}^{\infty}s_j(A)\right]=e^{|\mu||A|_1}.$$

Thus (1.2) is established.

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[2] For the definition of the genus of an entire function, see the book by B. Ja. Levin [1], Chapter I, §3.

Since, according to the definition of $D_A(\mu)$, it is an entire function of genus zero, it will at the same time be of minimal exponential type, i.e.,

$$\ln|D_A(\mu)| = o(|\mu|) \qquad (\mu \to \infty).$$

This follows at once from the relation

$$|D_A(\mu)| \leq \prod_{j=1}^{N} (1 + |\mu|\,|\lambda_j(A)|) \exp\left[|\mu| \sum_{j=N+1}^{\nu(A)} |\lambda_j(A)|\right].$$

2. The logarithmic derivative of the function $D_A(\mu)$ obviously has the form

$$\frac{D'_A(\mu)}{D_A(\mu)} = -\sum_{j=1}^{\nu(A)} \frac{\lambda_j}{1-\mu\lambda_j} \quad (\lambda_j = \lambda_j(A);\ j = 1, 2, \cdots, \nu(A));$$

consequently

$$\frac{D'_A(\mu)}{D_A(\mu)} = -\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\nu(A)} \lambda_j^k\right) \mu^{k-1} \quad (|\mu| < |\lambda_1|^{-1})$$

or

$$\frac{D'_A(\mu)}{D_A(\mu)} = -\operatorname{sp}(A(I-\mu A)^{-1}). \tag{1.3}$$

Analytically extending the right and left sides of (1.3), we conclude that (1.3) holds for all F-regular points μ of A.

We shall call the complex number μ an *F-regular point* (regular in the sense of Fredholm) of the operator A, if the operator $I - \mu A$ has an inverse.

It follows from (1.3) that

$$D_A(\mu) = \exp\left[-\int_0^\mu \operatorname{sp} A(\mu)\, d\mu\right]$$

for any F-regular point μ of the operator A, where $A(\mu) = A(I - \mu A)^{-1}$ is the Fredholm resolvent of the operator A, defined by

$$I + \mu A(\mu) = (I - \mu A)^{-1}.$$

Formula (1.3) enables us to show easily that in the case where the operator K ($\in \mathfrak{S}_1$) in $L_2(0,1)$ is defined by a continuous kernel $\mathscr{K}(t,s)$, the determinant $\det(I - \lambda K)$ coincides with the Fredholm determinant of this kernel. In fact, as is well known, one has for the Fredholm determinant $D(\lambda)$ of the kernel $\mathscr{K}(t,s)$ the equality

$$\int_0^1 \Gamma(s,s;\lambda)\, ds = -\frac{D'(\lambda)}{D(\lambda)},$$

where $\Gamma(t, s; \lambda)$ is the resolvent kernel. For small λ one has

$$\int_0^1 \Gamma(s, s; \lambda)\, ds = \sum_{n=1}^{\infty} \lambda^{n-1} \int_0^1 \mathscr{K}_n(s, s)\, ds$$

$$= \sum_{n=1}^{\infty} \lambda^{n-1} \operatorname{sp} K^n = \operatorname{sp}[K(I - \lambda K)^{-1}].$$

Hence by (1.3) we obtain

$$\frac{d}{d\lambda} \ln \det(I - \lambda K) = \frac{d}{d\lambda} \ln D(\lambda)$$

and

$$\det(I - \lambda K) = D(\lambda).$$

THEOREM 1.1 (On the continuous dependence of the determinant $D_A(\mu)$ on the operator A). *Let $A \in \mathfrak{S}_1$ and let F be an arbitrary closed bounded set of points of the complex plane. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for any operator $B \in \mathfrak{S}_1$ satisfying the inequality $|A - B|_1 < \delta$ one has*

$$\max_{\mu \in F} |D_A(\mu) - D_B(\mu)| < \epsilon.$$

PROOF. Let Γ be a simple rectifiable contour, consisting of F-regular points of the operator A, which encloses the set F and also the point $\mu = 0$.

By virtue of the maximum modulus principle, the theorem will be proved as soon as we prove the existence of $\delta > 0$ such that, for any $B \in \mathfrak{S}_1$,

$$\max_{\mu \in \Gamma} |D_A(\mu) - D_B(\mu)| < \epsilon$$

whenever $|B - A|_1 < \delta$.

Let us denote by L a simple rectifiable curve, consisting of F-regular points of the operator A, which connects the point $\mu = 0$ with some point of the contour Γ. Let Γ_μ ($\mu \in \Gamma \cup L$) denote the shortest path along the curves Γ and L which connects the point $\mu = 0$ to the point μ.

From the equality

$$(I - \mu B)^{-1} = (I - \mu A)^{-1} (I - \mu(B - A)\ (I - \mu A)^{-1})^{-1}$$

it follows that when the condition

$$|A - B|_1 < \min_{\mu \in \Gamma \cup L} [|\mu|\ |(I - \mu A)^{-1}|]^{-1}$$

is fulfilled, all points μ of the curves $\Gamma \cup L$ are F-regular points of the operator B, and

$$\max_{\mu \in \Gamma \cup L} |(I - \mu B)^{-1}| < C \max_{\mu \in \Gamma \cup L} |(I - \mu A)^{-1}|, \tag{1.4}$$

where C is a constant, depending only upon the curves Γ and L and the operator A. Since

$$A(\mu) - B(\mu) = (I - \mu A)^{-1}(A - B)\,(I - \mu B)^{-1},$$

we have

$$|A(\mu) - B(\mu)|_1 \leq |(I - \mu A)^{-1}|\,|B - A|_1\,|(I - \mu B)^{-1}| \qquad (\mu \in \Gamma \cup L). \tag{1.5}$$

It follows from (1.4) and (1.5) that for any $\eta > 0$ there exists $\delta > 0$ such that

$$\left|\int_{\Gamma_{\mu_0}} \mathrm{sp}\,(A(\mu) - B(\mu))\,d\mu\right| \leq \int_{\Gamma_{\mu_0}} |A(\mu) - B(\mu)|_1\,d\mu < \eta \qquad (\mu_0 \in \Gamma \cup L) \tag{1.6}$$

whenever $|A - B|_1 < \delta$. By choosing η sufficiently small, we can obviously arrange that from (1.6) will follow

$$\begin{aligned}&|D_A(\mu_0) - D_B(\mu_0)|\\ &\quad = \left|D_A(\mu_0)\left(1 - \exp\left[\int_{\Gamma_{\mu_0}} \mathrm{sp}(A(\mu) - B(\mu))\,d\mu\right]\right)\right| < \epsilon\\ &\qquad\qquad (\mu_0 \in \Gamma \cup L).\end{aligned}$$

The theorem is proved.

COROLLARY 1.1. *Let $A \in \mathfrak{S}_1$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that, for every operator $B \in \mathfrak{S}_1$ satisfying the inequality $|A - B|_1 < \delta$,*

$$|\det(I - A) - \det(I - B)| < \epsilon.$$

3. Corollary 1.1 enables us to state a number of rules for calculating the determinant $\det(I - A)$.

2. *Let $A \in \mathfrak{S}_1$ and let $\{\phi_n\}_1^\infty$ be an arbitrary othonormal basis of the space $\mathfrak{H}$; then*

$$\det(I - A) = \lim_{n \to \infty} \det\|\delta_{jk} - (A\phi_j, \phi_k)\|_1^n.$$

This expression can be written in the form

$$\det\|\delta_{jk} - (A\phi_j, \phi_k)\|_1^\infty.$$

In fact, $\det\|\delta_{jk} - (A\phi_j, \phi_k)\|_1^n$ is the determinant of the operator

$$I - A_n = I - \sum_{j=1}^n (\cdot, \phi_j)\,A\phi_j,$$

and since by Theorem III.6.3 the sequence of finite-dimensional operators $A_n\,(n = 1, 2, \cdots)$ tends to the operator A in the norm of the space $\mathfrak{S}_1$, we have

$$\lim_{n\to\infty} \det(I - A_n) = \det(I - A).$$

This result can be supplemented by a result which in certain cases makes it possible to simplify the calculation of $\det(I - A)$.

3. *Let $A \in \mathfrak{S}_1$ and let $\{\phi_j\}_1^\omega$ $(\omega \leqq \infty)$ be an orthonormal basis of the closure of $\mathfrak{R}(A)$. Then*

$$\det(I - A) = \det\| \delta_{jk} - (A\phi_j, \phi_k) \|_1^\omega. \tag{1.7}$$

In fact, let $\{\psi_j\}_1^{\omega_1}$ be an orthonormal basis of $\mathfrak{H}_1 = \mathfrak{H} \ominus \overline{\mathfrak{R}(A)}$. We combine the ϕ_j $(j = 1, 2, \cdots, \omega)$ and the ψ_j $(j = 1, 2, \cdots, \omega_1)$ into one sequence $\{\chi_j\}_1^\infty$; then

$$\det(I - A) = \det\| \delta_{jk} - (A\chi_j, \chi_k) \|_1^\infty = \lim_{n\to\infty} \det\| \delta_{jk} - (A\chi_j, \chi_k) \|_1^n.$$

But if $\chi_k = \psi_{p_k}$ for some k, then, bearing in mind that $\psi_{p_k} \in \mathfrak{R}(A)^\perp$, we will have

$$(A\chi_j, \chi_k) = 0 \qquad (j = 1, 2, \cdots),$$

from which it is easy to obtain (1.7).

4. *Let*

$$A = \sum_{j=1}^\infty (\cdot, \phi_j)\, \psi_j$$

be one of the possible representations of the operator $A \in \mathfrak{S}_1$, where $\{\phi_j\}$ and $\{\psi_j\}$ are arbitrary systems of vectors satisfying the condition

$$\sum_{j=1}^\infty |\phi_j|\, |\psi_j| < \infty.$$

Then

$$\det(I - A) = \lim_{n\to\infty} \det\| \delta_{jk} - (\psi_j, \phi_k) \|_1^n \qquad (= \det\| \delta_{jk} - (\psi_j, \phi_k) \|_1^\infty).$$

In fact, the operators

$$A_n = \sum_{j=1}^n (\cdot, \phi_j)\, \psi_j$$

tend to the operator A in the norm of $\mathfrak{S}_1$:

$$\lim_{n\to\infty} |A - A_n|_1 \leqq \lim_{n\to\infty} \sum_{j=n+1}^\infty |\phi_j|\, |\psi_j| = 0.$$

Consequently the result will be proved as soon as we establish the

equalities

$$\det(I - A_n) = \det\| \delta_{jk} - (\psi_j, \phi_k) \|_1^n \qquad (n = 1, 2, \cdots).$$

The verification of these equalities is left to the reader.

4. In the sequel the following results will also be used.

5. *Let $A \in \mathfrak{S}_\infty$ and $B \in \mathfrak{R}$ be such that $AB \in \mathfrak{S}_1$ and $BA \in \mathfrak{S}_1$. Then*

$$\det(I - AB) = \det(I - BA). \tag{1.8}$$

Let $s_j = s_j(A)$ $(j = 1, 2, \cdots)$ be the sequence of s-numbers of the operator A, and $\{\phi_j\}_1^\omega$ and $\{\psi_j\}_1^\omega$ corresponding orthonormal Schmidt systems, so that

$$A\phi_j = s_j \psi_j, \quad A^*\psi_j = s_j \phi_j \qquad (j = 1, 2, \cdots).$$

Since the system $\{\psi_j\}_1^\omega$ is complete in $\mathfrak{R}(A)$, on the basis of result 3 we can write

$$\begin{aligned} \det(I - AB) &= \det\| \delta_{jk} - (AB\psi_j, \psi_k) \|_1^\omega = \det\| \delta_{jk} - (B\psi_j, A^*\psi_k) \|_1^\omega \\ &= \det\| \delta_{jk} - (B\psi_j, \phi_k)\, s_k \|_1^\omega. \end{aligned} \tag{1.9}$$

Similarly, since $\{\phi_j\}_1^\omega$ is complete in $\mathfrak{R}(A^*)$, we have

$$\det(I - A^*B^*) = \det\| \delta_{jk} - (B^*\phi_j, \psi_k)\, s_k \|_1^\omega = \det\| \delta_{jk} - (\phi_j, B\psi_k)\, s_k \|_1^\omega.$$

Hence

$$\det(I - BA) = \overline{\det(I - A^*B^*)} = \det\| \delta_{jk} - (B\psi_k, \phi_j)\, s_k \|_1^\omega.$$

The last determinant is obtained from the determinant in (1.9) by multiplying its jth row by s_j and dividing its kth column by s_k $(j, k = 1, 2, \cdots)$ and then transposing. Therefore (1.9) is proved.

6. *Let $C \in \mathfrak{S}_1$ and let S be a bounded invertible operator. Then*

$$\det(I - C) = \det(I - SCS^{-1}).$$

This is a special case of result 5, obtained for $A = CS^{-1}$ and $B = S$.

7. *For any operators $A, B \in \mathfrak{S}_1$ we have*

$$\det[(I - A)(I - B)] = \det(I - A)\det(I - B). \tag{1.10}$$

In fact, denoting by $\{\phi_j\}_1^\infty$ an arbitrary orthonormal basis of the space $\mathfrak{H}$ and by P_n $(n = 1, 2, \cdots)$ the orthogonal projector on the linear hull of the vectors $\{\phi_j\}_1^n$, by virtue of Corollary 1.1 we obtain

$$\det[(I - A)(I - B)] = \lim_{n \to \infty} \det[(I - P_nAP_n)(I - P_nBP_n)]. \tag{1.11}$$

Since

$$\begin{aligned} &\det[(I - P_nAP_n)(I - P_nBP_n)] \\ &\qquad = \det\| ((I - P_nAP_n)(I - P_nBP_n)\phi_j, \phi_k) \|_1^n, \end{aligned} \tag{1.12}$$

$$\begin{aligned}(1.13)\qquad &\|((I - P_nAP_n)(I - P_nBP_n)\phi_j, \phi_k)\|_1^n \\ &= \|((I - P_nAP_n)\phi_j, \phi_k)\|_1^n \cdot \|((I - P_nBP_n)\phi_j, \phi_k)\|_1^n\end{aligned}$$

and the determinants of the matrices on the right side of the last equality tend as $n \to \infty$ to $\det(I - A)$ and $\det(I - B)$ respectively, (1.10) follows from (1.11)—(1.13).

8. *Let $A(\mu)$ be an operator-function with values in $\mathfrak{S}_1$, and holomorphic in some region. Then the determinant* $\det(I - A(\mu))$ *is holomorphic in the same region.*

In fact, let $\{\phi_j\}_1^\infty$ and P_n $(n = 1, 2, \cdots)$ be as in the proof of result 7. Then at every point μ of the region G in question

$$\Delta(\mu) = \det(I - A(\mu)) = \lim_{n\to\infty} \det(I - P_nA(\mu)P_n) = \lim_{n\to\infty} \Delta_n(\mu).$$

The determinants $\Delta_n(\mu)$ $(n = 1, 2, \cdots)$ are holomorphic functions in G. Since

$$|\Delta_n(\mu)| = |\det(I - P_nA(\mu)P_n)| \leqq \exp|A(\mu)|_1 \qquad (n = 1, 2, \cdots),$$

the functions $\Delta_n(\mu)$ $(n = 1, 2, \cdots)$ are uniformly bounded on any bounded closed part of the region G. Hence by a well-known theorem of the theory of functions, their limit is a holomorphic function in G.

9. *Under the hypotheses of result* 8, *the formula*

$$(1.14)\qquad \frac{d}{d\mu}\ln\det(I - A(\mu)) = -\operatorname{sp}\left[(I - A(\mu))^{-1}\frac{dA(\mu)}{d\mu}\right]$$

is valid for all points μ at which the operator $I - A(\mu)$ has a bounded inverse.

We shall first prove formula (1.14) under the assumption that the space $\mathfrak{H}$ is finite-dimensional. Let $\{\phi_j\}_1^n$ be an orthonormal basis of $\mathfrak{H}$ and

$$a_{jk}(\mu) = (A(\mu)\phi_k, \phi_j) \qquad (j, k = 1, 2, \cdots, n).$$

Then the left side of (1.14) can be represented in the form

$$(1.15)\qquad \frac{d}{d\mu}\ln\det(I - A(\mu)) = \frac{\Delta'(\mu)}{\Delta(\mu)} = \left[\sum_{k,j=1}^{n} \gamma_{kj}(\mu)\frac{d}{d\mu}a_{kj}(\mu)\right]\Delta^{-1}(\mu),$$

where $\gamma_{kj}(\mu)$ is the cofactor of the element $\delta_{kj} - a_{kj}(\mu)$ in the determinant $\Delta(\mu) = \det\|\delta_{kj} - a_{kj}(\mu)\|_1^n$. As is easily seen, the right side of (1.15) coincides with the right side of (1.14).

It follows from the proof that (1.15) holds if we replace the operator-function $A(\mu)$ in it by the operator-function $P_nA(\mu)P_n$, where P_n is the orthoprojector onto the linear hull of the first n vectors of some orthonormal basis $\{\phi_j\}_1^\infty$ of the space $\mathfrak{H}$. Passing to the limit $n \to \infty$ in the equality thus obtained, we get (1.14).

We remark that the formula (1.3) is a special case of formula (1.14);

namely, it is obtained from the latter by putting $A(\mu) = \mu A$ $(A \in \mathfrak{S}_1)$.

Formula (1.14) can be obtained as an immediate corollary of the general formula[3)]

$$\frac{d}{d\mu}\,\mathrm{sp}\{F(A(\mu))\} = \mathrm{sp}\left\{F'(A(\mu))\frac{dA(\mu)}{d\mu}\right\}, \tag{1.16}$$

where $F(z)$ is any scalar function, holomorphic in some region G_F of the complex plane, $\sigma(A(\mu)) \in G_F$ for all $\mu \in G$, and $F(0) = 0$.

In fact, if $A \in \mathfrak{S}_1$ and $\det(I - A) \neq 0$, then we will have

$$\ln\det(I - A) = \mathrm{sp}\ln(I - A), \tag{1.17}$$

and therefore (1.14) can be rewritten in the form

$$\frac{d}{d\mu}\,\mathrm{sp}\ln(I - A(\mu)) = -\,\mathrm{sp}\left\{(I - A(\mu))^{-1}\frac{dA(\mu)}{d\mu}\right\}.$$

This is a special case of formula (1.16), corresponding to $F(z) = \ln(1 - z)$.

We shall shortly give a derivation of formula (1.16). First we make some preliminary remarks.

First of all, we recall (see Riesz and Sz.-Nagy [1]) that for $A \in \mathfrak{R}$ and $F(z)$ a function holomorphic in some region G_F containing the spectrum $\sigma(A)$ of the operator A, the operator $F(A)$ is defined by means of an integral:

$$F(A) = -\frac{1}{2\pi i}\int_\gamma F(z)(A - zI)^{-1}dz, \tag{1.18}$$

where γ is an oriented simple or multiple rectifiable contour which encloses (on the left) some region which is contained in G_F and which contains the spectrum $\sigma(A)$. As is known, the operator $F(A)$ does not depend upon the choice of γ.

In particular, if $A \in \mathfrak{S}_\infty$, this definition includes within itself the requirement that the point $\lambda = 0$ belongs to G_F. If $F(0) = 0$, then, using the fact that $(A - zI)^{-1} + z^{-1}I = z^{-1}A(A - zI)^{-1}$, we easily obtain from (1.18):

$$F(A) = -\frac{1}{2\pi i}\int_\gamma F(z)A(A - zI)^{-1}\frac{dz}{z}. \tag{1.19}$$

[3)] This formula is known; in the book by Dixmier [1] it is derived for the case of an operator-function $A(\mu)$ defined on an interval $(a \leq \mu \leq b)$ with values in $\mathfrak{S}_1$, which is continuously differentiable in $\mathfrak{S}_1$. The derivation of the formula in our case is completely analogous.

If $A \in \mathfrak{S}$, where $\mathfrak{S}$ is some s.n. ideal, then $A(A - zI)^{-1}$ will be a continuous operator-function on γ with values in $\mathfrak{S}$. Therefore the integral in (1.19) will always have as its value an operator from the same ideal $\mathfrak{S}$.

Thus the supplementary condition $F(0) = 0$ ensures that if $A \in \mathfrak{S}$, then also $F(A) \in \mathfrak{S}$. It is not particularly difficult to show that if the operator A has the spectrum of eigenvalues $\{\lambda_j(A)\}_1^{\nu(A)}$, then the operator $F(A)$ has the spectrum of eigenvalues $\{F(\lambda_j(A))\}_1^{\nu(A)}$. Following these clarifications, and recalling the definition (1.1) of the determinant $\det(I - A)$ for $A \in \mathfrak{S}_1$, we can prove (1.17) for that branch $F(z) = \ln(1 - z)$ of the multi-valued function $\mathrm{Ln}(1 - z)$ which is holomorphic in some region containing $\sigma(A)$ and is singled out by the condition $F(0) = 0$.

We now present the proof of formula (1.16).

Choosing an arbitrary point $\mu_0 \in G$, we construct an oriented rectifiable contour γ which encircles (on the left) some region G_γ which lies in G_F and contains $\sigma(A(\mu_0))$. Then for all μ from some neighborhood $S(\mu_0)$ of the point μ_0 we will have $\sigma(A(\mu)) \in G_\gamma$, and consequently

$$F(A(\mu)) = -\frac{1}{2\pi i}\int_\gamma F(z)\, C(\mu, z)\, dz, \tag{1.20}$$

where

$$\begin{aligned} C(\mu, z) &= z^{-1}A(\mu)(A(\mu) - zI)^{-1} \\ &= (A(\mu) - zI)^{-1} + z^{-1}I \qquad (z \in \gamma,\ \mu \in G_\gamma). \end{aligned} \tag{1.21}$$

Since for any fixed $\mu \in G_\gamma$ the function $C(\mu, z)$ is a continuous operator-function of $z \in \gamma$ with values in $\mathfrak{S}_1$, it follows from (1.20) that

$$\operatorname{sp} F(A(\mu)) = -\frac{1}{2\pi i}\int_\gamma F(z)\operatorname{sp}\{C(\mu, z)\}\, dz. \tag{1.22}$$

According to (1.21) we have

$$\frac{C(\mu, z) - C(\mu_0, z)}{\mu - \mu_0} = (A(\mu_0) - zI)^{-1}\frac{A(\mu) - A(\mu_0)}{\mu - \mu_0}(A(\mu) - zI)^{-1}.$$

Since by hypothesis the limit

$$\frac{dA}{d\mu_0} = \lim_{\mu \to \mu_0}\frac{A(\mu) - A(\mu_0)}{\mu - \mu_0}$$

exists in the metric of $\mathfrak{S}_1$, it is not hard to see that the limit

$$\frac{dC(\mu_0,z)}{d\mu_0} = \lim_{\mu\to\mu_0} \frac{C(\mu,z) - C(\mu_0,z)}{\mu-\mu_0}$$
$$= (A(\mu_0) - zI)^{-1}\frac{dA}{d\mu_0}(A(\mu_0) - zI)^{-1}$$

exists, uniformly relative to $z \in \gamma$, in the metric of $\mathfrak{S}_1$. Therefore the limit

$$\frac{d}{d\mu_0}\operatorname{sp}\{C(\mu_0,z)\} = \operatorname{sp}\left\{(A(\mu_0) - zI)^{-1}\frac{dA}{d\mu_0}(A(\mu_0) - zI)^{-1}\right\}$$
$$= \operatorname{sp}\left\{(A(\mu_0) - zI)^{-2}\frac{dA}{d\mu_0}\right\}$$
$$= -\frac{d}{dz}\operatorname{sp}\left\{(A(\mu_0) - zI)^{-1}\frac{dA}{d\mu_0}\right\}$$

exists uniformly in $z \in \gamma$. Recalling (1.22), we find that

$$\frac{d}{d\mu_0}\operatorname{sp}\{F(A(\mu_0))\} = \frac{1}{2\pi i}\int_\gamma F(z)\frac{d}{dz}\operatorname{sp}\left\{(A(\mu_0) - zI)^{-1}\frac{dA}{d\mu_0}\right\}dz$$
$$= -\frac{1}{2\pi i}\int_\gamma F'(z)\operatorname{sp}\left\{(A(\mu_0) - zI)^{-1}\frac{dA}{d\mu_0}\right\}dz.$$

The last integral is obtained by integrating by parts, and can be further transformed to the form

$$\operatorname{sp}\left\{-\frac{1}{2\pi i}\int_\gamma F'(z)\cdot(A(\mu_0) - zI)^{-1}dz\frac{dA}{d\mu_0}\right\} = \operatorname{sp}\left\{F'(A(\mu_0))\frac{dA}{d\mu_0}\right\},$$

which concludes the derivation of (1.16).

§2. Regularized characteristic determinants for the operators from $\mathfrak{S}_p$

1. With every operator $A \in \mathfrak{S}_p$, where p is a positive integer, we associate the number

$$\overset{(p)}{\det}(I - A) = \prod_{j=1}^{\nu(A)}\left[(1 - \lambda_j(A))\exp\sum_{k=1}^{p-1}\frac{1}{k}\lambda_j^k(A)\right],$$

which we shall call the *regularized determinant* of the operator $I - A$. For a Volterra operator A we put $\det^{(p)}(I - A) = 1$.

We shall call the determinant

$$(2.1)\quad \overset{(p)}{\det}(I - \mu A) = \prod_{j=1}^{\nu(A)} \left[(1 - \lambda_j(A)\mu) \exp \sum_{k=1}^{p-1} \frac{1}{k} \lambda_j^k(A)\mu^k \right] \qquad (A \in \mathfrak{S}_p)$$

the *regularized characteristic determinant* of the operator A, and shall denote it by $D_A^{(p)}(\mu)$.

If $A \in \mathfrak{S}_1$ then obviously

$$\overset{(p)}{D}_A(\mu) = D_A(\mu) \exp \sum_{k=1}^{p-1} \frac{1}{k} \operatorname{sp} A^k \mu^k.$$

From the representation (2.1) of the determinant $D_A^{(p)}(\mu)$, we see that it is an entire function of genus $p-1$ and, consequently, an entire function of at most order p and minimal type, i.e.,

$$\ln |\overset{(p)}{D}_A(\mu)| = o(|\mu|^p) \qquad (\mu \to \infty).$$

The logarithmic derivative of the function $D_A^{(p)}(\mu)$ has the form

$$\frac{d}{d\mu} \ln \overset{(p)}{D}_A(\mu) = -\sum_{j=1}^{\infty} \sum_{k=p}^{\infty} \lambda_j^k \mu^{k-1} \qquad (|\mu| < |\lambda_1|^{-1});$$

consequently

$$\frac{d}{d\mu} \ln \overset{(p)}{D}_A(\mu) = -\operatorname{sp}\{\mu^{p-1} A^p (I - \mu A)^{-1}\}.$$

This equality (which is analogous to (1.3)) holds for all complex μ which are F-regular points of the operator A. Thus for any F-regular point μ of A one has the equality

$$\overset{(p)}{D}_A(\mu) = \exp \left[-\int_0^\mu \operatorname{sp}[\mu^{p-1} A^p (I - \mu A)^{-1} \right] d\mu.$$

This representation of the function $D_A^{(p)}(\mu)$ enables us to prove the following generalization of Theorem 1.1.

THEOREM 2.1. *Let $A \in \mathfrak{S}_p$, where p is a positive integer, and let F be an arbitrary closed bounded set. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for any operator $B \in \mathfrak{S}_p$*

$$\max_{\mu \in F} |\overset{(p)}{D}_A(\mu) - \overset{(p)}{D}_B(\mu)| < \epsilon$$

whenever

$$|A - B|_p < \delta.$$

The proof of this theorem on the continuous dependence of the regularized determinant upon the operator A is similar to the proof of Theorem 1.1, and is omitted.

2. If $A \in \mathfrak{S}_2$, then by definition

$$\overset{(2)}{D}_A(\mu) = \prod_{j=1}^{\nu(A)} [(1 - \lambda_j(A)\,\mu)\, e^{\mu\lambda_j(A)}].$$

For convenience this determinant will henceforth be denoted by $\widetilde{D}_A(\mu)$, and the determinant $\det^{(2)}(I - A)$ $(A \in \mathfrak{S}_2)$, by $\widetilde{\det}(I - A)$.

1. *The determinant* $\widetilde{D}_A(\mu)$ $(A \in \mathfrak{S}_2)$ *admits the precise bound*

$$|\widetilde{D}_A(\mu)| \leqq \exp\left[\frac{1}{2}|\mu|^2 \mathrm{sp}(A^*A)\right]. \tag{2.2}$$

In fact,

$$|\widetilde{D}_A(\mu)|^2 = \prod_{j=1}^{\nu(A)} |1 - \mu\lambda_j|^2 e^{2\mathrm{Re}(\mu\lambda_j)} \quad (\lambda_j = \lambda_j(A)),$$

$$|\widetilde{D}_A(\mu)|^2 = \prod_{j=1}^{\nu(A)} (1 - 2\mathrm{Re}(\mu\lambda_j) + |\mu\lambda_j|^2)\, e^{2\mathrm{Re}(\mu\lambda_j)}. \tag{2.3}$$

Since $1 + x \leqq e^x$, (2.3) implies the inequality

$$|\widetilde{D}_A(\mu)|^2 \leqq \exp\left(|\mu^2| \sum_{j=1}^{\nu(A)} |\lambda_j|^2\right).$$

Taking into account that

$$\sum_{j=1}^{\nu(A)} |\lambda_j|^2 \leqq \sum_{j=1}^{\infty} s_j^2(A) = \mathrm{sp}(A^*A),$$

we obtain (2.2).

The bound (2.2) is precise. One can see this by considering the sequence of operators

$$K_n = \sqrt{\frac{l}{2n}}\left[\sum_{j=1}^{n} (\cdot, \phi_j)\, \phi_j - \sum_{j=n+1}^{2n} (\cdot, \phi_j)\, \phi_j\right] \quad (n = 1, 2, \cdots),$$

where $\{\phi_j\}_1^\infty$ is some orthonormal system. Indeed, it is easy to calculate that

$$\widetilde{D}_{K_n}(\mu) = \left(1 - \mu\sqrt{\frac{l}{2n}}\right)^n \exp\left[\sqrt{\frac{nl}{2}}\,\mu\right]\left(1 + \mu\sqrt{\frac{l}{2n}}\right)^n \exp\left[-\sqrt{\frac{nl}{2}}\,\mu\right]$$
$$= \left(1 - \mu^2\frac{l}{2n}\right)^n.$$

From this we obtain

$$\lim_{n\to\infty} \tilde{D}_{K_n}(\mu) = e^{-\mu^2 l/2}.$$

It remains to note that $l = \operatorname{sp}(K_n^* K_n)$ $(n = 1, 2, \cdots)$ and to consider pure imaginary μ.

2. *Let $A \in \mathfrak{S}_2$ and let $\{\phi_j\}_1^\infty$ be an arbitrary orthonormal basis of the space $\mathfrak{H}$. Then*

$$\widetilde{\det}(I - A) = \lim_{n\to\infty}\left[\det\|\delta_{jk} - (A\phi_j, \phi_k)\|_1^n \exp\sum_{j=1}^{n}(A\phi_j, \phi_j)\right]. \tag{2.4}$$

In fact,

$$\widetilde{\det}(I - A_n) = \det\|\delta_{jk} - (A\phi_j, \phi_k)\|_1^n \exp\sum_{j=1}^{n}(A\phi_j, \phi_j) \quad (n = 1, 2, \cdots),$$

where

$$A_n = \sum_{j=1}^{n}(\cdot, \phi_j)A\phi_j \qquad (n = 1, 2, \cdots).$$

Since the sequence of operators $\{A_n\}_1^\infty$ tends to the operator A in the norm of $\mathfrak{S}_2$, we have according to Theorem 2.1

$$\widetilde{\det}(I - A) = \lim_{n\to\infty}\det(I - A_n).$$

3. *If $A, B \in \mathfrak{S}_2$, then for the operator $(I - C) = (I - A)(I - B)$ we have*

$$\widetilde{\det}(I - C)e^{\operatorname{sp}AB} = \widetilde{\det}(I - A)\widetilde{\det}(I - B). \tag{2.5}$$

Let $\{\phi_j\}_1^\infty$ be an orthonormal basis of the space $\mathfrak{H}$, P_n the orthogonal projector onto the linear hull of the vectors $\{\phi_j\}_1^n$, and $A_n = P_nAP_n$, $B_n = P_nBP_n$ $(n = 1, 2, \cdots)$. Then

$$\widetilde{\det}(I - A_n)\widetilde{\det}(I - B_n) = \det[(I - A_n)(I - B_n)]e^{\operatorname{sp}(A_n + B_n)},$$

and consequently

$$\widetilde{\det}(I - A_n)\widetilde{\det}(I - B_n) = \widetilde{\det}[(I - A_n)(I - B_n)]e^{\operatorname{sp}A_nB_n}.$$

Passing to the limit $n \to \infty$, we obtain (2.5).

3. The theory of infinite determinants developed by von Koch [1, 2] enables us to indicate a more profound approach to the concept of the determinant $\det(I - A)$ and the regularized determinant $\widetilde{\det}(I - A)$. We shall present without proof a number of results from this theory.

An infinite matrix $I - \mathscr{A} = \|\delta_{jk} - a_{jk}\|_1^\infty$, consisting of complex numbers, is called a matrix with an *absolutely convergent determinant*, if the series

$$\sum |a_{j_1k_1}a_{j_2k_2}\cdots a_{j_nk_n}|$$

converges, where the sum extends over all pairs of systems of indices $(j_1,j_2,\cdots,j_n)$ and $(k_1,k_2,\cdots,k_n)$ which differ from each other only by a permutation, and $j_1<j_2<\cdots<j_n$ $(n=1,2,\cdots)$.

If the matrix $I-\mathscr{A}$ is a matrix with an absolutely convergent determinant, then the limit

$$\lim_{n\to\infty}\det\|\delta_{jk}-a_{jk}\|_1^n=\det(I-\mathscr{A})$$

exists and is called the determinant of the matrix $I-\mathscr{A}$.

If the matrix $I-\mathscr{A}$ has an absolutely convergent determinant, its determinant can be represented in the form

$$\det(I-\mathscr{A})$$

$$=1-\sum_{j=1}^{\infty}a_{jj}+\frac{1}{2!}\sum_{j,k=1}^{\infty}\begin{vmatrix}a_{jj}&a_{kj}\\a_{jk}&a_{kk}\end{vmatrix}-\frac{1}{3!}\sum_{j,k,r=1}^{\infty}\begin{vmatrix}a_{jj}&a_{kj}&a_{rj}\\a_{jk}&a_{kk}&a_{rk}\\a_{jr}&a_{kr}&a_{rr}\end{vmatrix}+\cdots, \tag{2.6}$$

where the series in the right side of (2.6) will remain convergent if the numbers a_{jk} $(j,k=1,2,\cdots)$ are replaced by their moduli and if all the terms obtained by expanding the determinants in (2.6) are taken with a plus sign.

We shall call the matrix $I-\mathscr{A}$ a *von Koch matrix* if the conditions

$$\sum_{j=1}^{\infty}|a_{jj}|<\infty,\qquad \sum_{j,k=1}^{\infty}|a_{jk}|^2<\infty \tag{2.7}$$

are fulfilled. For such matrices one has the following results, which were established by von Koch [1, 2].

I. *Every von Koch matrix has an absolutely convergent determinant. If the elements of a von Koch matrix are functions of some parameter* μ; $a_{jk}=a_{jk}(\mu)$ $(j,k=1,2,\cdots)$ *and the series in* (2.7) *converge uniformly in the domain of the parameter* μ, *then as* $n\to\infty$ *the determinant* $\det\|\delta_{jk}-a_{jk}(\mu)\|_1^n$ *tends to the determinant* $\det(I-\mathscr{A}(\mu))$ *uniformly with respect to* μ, *over the domain of* μ.

II. *If the matrices* $I-\mathscr{A}$ *and* $I-\mathscr{B}$ *are von Koch matrices, then their product* $I-\mathscr{C}=(I-\mathscr{A})(I-\mathscr{B})$ *is a von Koch matrix, and*

$$\det(I-\mathscr{C})=\det(I-\mathscr{A})\det(I-\mathscr{B}).$$

Let A be a completely continuous operator, and let $\{\phi_j\}_1^\infty$ be an orthonormal basis of $\mathfrak{H}$. As is known, there corresponds to the operator A in the basis $\{\phi_j\}_1^\infty$ an infinite matrix

$$A_\phi=\|a_{jk}\|_1^\infty;\quad a_{jk}=(A\phi_k,\phi_j)\quad(j,k=1,2,\cdots).$$

The second condition in (2.7) will be fulfilled by the matrix A_ϕ if and only if $A \in \mathfrak{S}_2$. If $A \in \mathfrak{S}_1$, both conditions in (2.7) will be fulfilled by the matrix A_ϕ, and consequently the von Koch determinant $\det(I - A_\phi)$ will be meaningful. Recalling the result I, we conclude thet for $A \in \mathfrak{S}_1$

$$\det(I - A) = \det(I - A_\phi).$$

If A is an arbitrary operator from $\mathfrak{S}_2$, then generally speaking the first of conditions (2.7) will not be fulfilled by the matrix A_ϕ, and consequently the matrix A_ϕ will certainly not have an absolutely convergent von Koch determinant. However, if the matrix A_ϕ is 'regularized', replacing it by the matrix

$$\tilde{A}_\phi = \| \tilde{a}_{jk} \|_1^\infty,$$

where $\tilde{a}_{jk}$ is defined by the relation

$$\delta_{jk} - \tilde{a}_{jk} = (\delta_{jk} - a_{jk})\, e^{a_{jj}} \qquad (j, k = 1, 2, \cdots),$$

then the conditions (2.7) will once again be fulfilled. In fact, since $a_{jj} \to 0$ as $j \to \infty$,

$$\tilde{a}_{jj} = 1 - e^{a_{jj}}(1 - a_{jj}) = O(|a_{jj}|^2) \qquad (j \to \infty)$$

and

$$|\tilde{a}_{jk}| \leqq c\,|a_{jk}| \qquad (j, k = 1, 2, \cdots).$$

It is easily seen that

$$\tilde{D}_A(\mu) = \det(I - (\widetilde{\mu A_\phi})).$$

Starting from just this definition of the characteristic determinant, Hille and Tamarkin [1] obtained the bound (2.2), using Hadamard's well-known inequality.

We remark that for integral operators in $L_2(a, b)$ with a Hilbert-Schmidt kernel, the bound (2.2) for the Hilbert-regularized Fredholm determinant was obtained by Carleman [2]. He also determined that this bound is precise.

§3. Perturbation determinants

1. Let A and B be bounded linear operators acting in $\mathfrak{H}$, with $A - B \in \mathfrak{S}_1$.

If the point μ is an F-regular point of the operator A, then

$$(I - \mu B)(I - \mu A)^{-1} = I - \mu(B - A)(I - \mu A)^{-1}, \tag{3.1}$$

where $\mu(B - A)(I - \mu A)^{-1} \in \mathfrak{S}_1$. Consequently the determinant

$$D_{B/A}(\mu) = \det[(I - \mu B)(I - \mu A)^{-1}]$$

has meaning; we shall call it the *perturbation determinant* of the operator A by the operator $T = B - A$.

By result 8 of §1, the perturbation determinant $D_{B/A}(\mu)$ is a holomorphic function in the region consisting of all F-regular points of the operator A.

If $A, B \in \mathfrak{S}_1$, then obviously

$$D_{B/A}(\mu) = \frac{D_B(\mu)}{D_A(\mu)}. \tag{3.2}$$

1. *If $A, B \in \mathfrak{S}_2$, $A - B \in \mathfrak{S}_1$ and μ is an F-regular point of the operator A, then*

$$D_{B/A}(\mu) = \frac{\widetilde{D}_B(\mu)}{\widetilde{D}_A(\mu)} \exp[\mu \operatorname{sp}(A - B)]. \tag{3.3}$$

In fact, according to result 3 of §2 it follows from (3.1) that

$$\widetilde{D}_B(\mu)\, e^{\mu \operatorname{sp} MA} = \widetilde{\det}(I - M)\, \widetilde{D}_A(\mu),$$

where

$$M = \mu(B - A)(I - \mu A)^{-1}.$$

Since $M \in \mathfrak{S}_1$,

$$\widetilde{\det}(I - M) = \det(I - M)\, e^{\operatorname{sp} M}.$$

Thus

$$D_{B/A}(\mu) = \det(I - M) = \frac{\widetilde{D}_B(\mu)}{\widetilde{D}_A(\mu)} e^{-\operatorname{sp} M(I - \mu A)}.$$

Taking into account that $-M(I - \mu A) = \mu(A - B)$, we arrive at (3.3).

2. *Let $A, B, C \in \mathfrak{R}$. If the point μ is an F-regular point of the operators A and B, and if the operators $B - A$ and $C - B$ belong to $\mathfrak{S}_1$, then*

$$D_{C/B}(\mu)\, D_{B/A}(\mu) = D_{C/A}(\mu).$$

Indeed, this follows directly from the equality

$$[(I - \mu C)(I - \mu B)^{-1}][(I - \mu B)(I - \mu A)^{-1} = (I - \mu C)(I - \mu A)^{-1}$$

As a corollary of result 2 we obtain

3. *If the point μ is a common F-regular point of the operators A and B ($\in \mathfrak{R}$), and if $A - B \in \mathfrak{S}_1$, then*

$$D_{A/B}(\mu) = [D_{B/A}(\mu)]^{-1}.$$

Although in the general case $(A, B \notin \mathfrak{S}_1)$ the equality (3.2) is not meaningful, nevertheless the order[4] of the perturbation determinant at any point μ_0 $(\neq 0)$ is determined in a natural way by the multiplicities of $\lambda_0 = \mu_0^{-1}$ as an eigenvalue of the operators A and B.

4. *Suppose that the operators $A, B \in \mathfrak{R}$ satisfy the conditions*: a) $A - B \in \mathfrak{S}_1$, b) λ_0 *is a common normal point of A and B. Then the order of the perturbation determinant $D_{B/A}(\mu)$ at the point $\mu_0 = 1/\lambda_0$ is equal to the difference $\nu_{\lambda_0}(B) - \nu_{\lambda_0}(A)$, where $\nu_{\lambda_0}(A)$ and $\nu_{\lambda_0}(B)$ are the algebraic multiplicities of the number λ_0 for the operators A and B respectively.*

In fact, three cases can occur.

1) Let λ_0 be a regular point of the operators A and B. Then the point $\mu_0 = 1/\lambda_0$ is a regular point of the determinant $D_{B/A}(\mu)$ The operator $I + \mu_0(A - B)(I - \mu_0 A)^{-1} = (I - \mu_0 B)(I - \mu_0 A)^{-1}$ has a bounded inverse $(I - \mu_0 A)(I - \mu_0 B)^{-1}$, and so

$$D_{B/A}(\mu) = \det[(I - \mu_0 B)(I - \mu_0 A)^{-1}] \neq 0.$$

2) Let λ_0 be a regular point for the operator A and an eigenvalue of multiplicity $\nu_{\lambda_0}(B)$ $(\neq 0)$ of the operator B. We denote by $\mathfrak{L}$ the root subspace of the operator B corresponding to λ_0, and by $\mathfrak{N}$ the direct complement of $\mathfrak{L}$ in $\mathfrak{H}$ which is invariant with respect to B. Let us consider the operators $B_1 = B(I - P)$ and $B_2 = BP$, where P is the operator which projects $\mathfrak{H}$ onto $\mathfrak{L}$ parallel to $\mathfrak{N}$.

The point λ_0 is a regular point of the operator B_1. We denote by C_ϵ a neighborhood $|\lambda - \lambda_0| < \epsilon$ of the point λ_0 in which the operators $A - \lambda I$ and $B_1 - \lambda I$ are invertible.

Since the operator B_2 is finite-dimensional, result 2 can be applied to the triple of operators A, B_1 and B at the point $\mu = 1/\lambda$, i.e.

$$D_{B/A}(\mu) = D_{B/B_1}(\mu) D_{B_1/A}(\mu). \tag{3.4}$$

The operators B_1 and B_2 are orthogonal; hence

$$(I - \mu B)(I - \mu B_1)^{-1} = (I - \mu B_2)(I - \mu B_1)(I - \mu B_1)^{-1} = I - \mu B_2$$

and

$$D_{B/B_1}(\mu) = D_{B_2}(\mu).$$

[4] The order of an analytic function $f(z)$ at a point z_0 is the integer k $(= 0, \pm 1, \pm 2, \cdots)$ in the representation $f(z) = (z - z_0)^k g(z)$, where $g(z)$ is analytic at the point z_0 and $g(z_0) \neq 0$.

The determinant $D_{B_2}(\mu)$ has a zero of multiplicity $\nu_{\lambda_0}(B_2) = \nu_{\lambda_0}(B)$ at the point μ_0 $(= 1/\lambda_0)$.

Since, as was proved, $D_{B_1/A}(\mu_0) \neq 0$, it follows from (3.4) that the determinant $D_{B/A}(\mu)$ has a zero of multiplicity $\nu_{\lambda_0}(B)$ at the point μ_0.

3) We proceed to the proof of the result in the general case. As has been proved, the determinant $D_{A/B_1}(\mu)$ has a zero of multiplicity $\nu_{\lambda_0}(A)$ at the point μ_0, and so the determinant

$$D_{B_1/A}(\mu) = 1/D_{A/B_1}(\mu) \qquad (\lambda \in C_\epsilon;\ \lambda \neq \lambda_0)$$

has a pole of order $\nu_{\lambda_0}(A)$ at the point μ_0. It follows from (3.4) that the order of $D_{B/A}(\mu)$ at the point $\mu = \mu_0$ equals $\nu_{\lambda_0}(B) - \nu_{\lambda_0}(A)$. The result is proved.

2. The determinant $D_{B/A}(\mu)$ ought to be called the F-perturbation determinant (Fredholm) in contrast to the perturbation determinant $\Delta_{B/A}(\mu)$ defined by

$$\Delta_{B/A}(\lambda) = \det((B - \lambda I)(A - \lambda I)^{-1}) \qquad (B - A \in \mathfrak{S}_1).$$

Obviously,

$$\Delta_{B/A}(\lambda) = D_{B/A}(1/\lambda).$$

Result 4 assumes a more natural form for the determinant $\Delta_{B/A}(\lambda)$: the order of $\Delta_{B/A}(\lambda)$ at a normal point λ_0 of A and B equals $\nu_{\lambda_0}(B) - \nu_{\lambda_0}(A)$.

As we indicated in the introduction to this chapter, the concept of the perturbation determinants Δ and D can be extended to certain pairs of unbounded operators. It is usually more convenient, in the theory of perturbations, to use the determinant $\Delta_{B/A}(\lambda)$.

For completeness we present the following result, which plays an important role in the theory of perturbations (cf. M. G. Kreĭn [**6**]).

5. *Let $A, B \in \mathfrak{R}$ and $A - B \in \mathfrak{S}_1$. Then for any point μ which is F-regular for A and B, we have*

$$\frac{d}{d\mu} \ln D_{B/A}(\mu) = \operatorname{sp}(A(\mu) - B(\mu)), \tag{3.5}$$

where $A(\mu)$ and $B(\mu)$ are the Fredholm resolvents of the operators A and B. In particular, for μ of sufficiently small modulus one has the expansion

$$\frac{d}{d\mu} \ln D_{B/A}(\mu) = \sum_{j=0}^{\infty} \mu^j \operatorname{sp}(A^{j+1} - B^{j+1}). \tag{3.6}$$

Indeed, if μ is a common F-regular point of the operators A and B, then

(3.7) $$(I-\mu A)^{-1}-(I-\mu B)^{-1}=\mu(I-\mu B)^{-1}(A-B)(I-\mu A)^{-1}$$

and

(3.8) $$A(\mu)-B(\mu)=(I-\mu B)^{-1}(A-B)(I-\mu A)^{-1},$$

so that the right side of (3.5) is meaningful.

It follows from (3.7) that

$$(I-\mu B)^{-1}=(I-\mu A)^{-1}[I+\mu(A-B)(I-\mu A)^{-1}]^{-1}.$$

Inserting this expression for $(I-\mu B)^{-1}$ into the right side of (3.8) and taking the trace of both sides, we obtain

$$\operatorname{sp}(A(\mu)-B(\mu))=\operatorname{sp}\{(I+\mu(A-B)(I-\mu A)^{-1})^{-1}(A-B)(I-\mu A)^{-2}\}.$$

Since

$$\frac{d\mu(I-\mu A)^{-1}}{d\mu}=(I-\mu A)^{-1}+\mu A(I-\mu A)^{-2}=(I-\mu A)^{-2},$$

for all F-regular points μ of the operator A we will have

$$\frac{dC(\mu)}{d\mu}=(A-B)(I-\mu A)^{-2},$$

where $C(\mu)=\mu(A-B)(I-\mu A)^{-1}$.

Thus on the basis of result 9 of §1 we obtain

$$\operatorname{sp}(A(\mu)-B(\mu))=\operatorname{sp}\left\{(I+C(\mu))^{-1}\frac{dC_\mu}{d\mu}\right\}=\frac{d}{d\mu}\ln\det(I+C(\mu)).$$

The result is proved. It is easily seen that formula (3.5) for the determinant $\Delta_{B/A}(\lambda)$ can be rewritten in the form

$$\frac{d}{d\lambda}\ln\Delta_{B/A}(\lambda)=\operatorname{sp}((B-\lambda I)^{-1}-(A-\lambda I)^{-1}).$$

This formula can be generalized to certain pairs of unbounded operators (cf. M. G. Kreĭn [6] and S. T. Kuroda [2]).

§4. A lemma on the growth of the perturbation determinant of a dissipative operator

1. A linear operator A, acting in $\mathfrak{H}$ and having domain $\mathfrak{D}(A)$, is said to be *dissipative* if

$$\operatorname{Im}(Af,f)\geqq 0 \qquad \text{for } f\in\mathfrak{D}(A).$$

For a bounded operator A (defined on all of $\mathfrak{H}$) the condition of dissipativeness is equivalent to the condition that its imaginary com-

ponent $A_{\mathscr{I}} = (A - A^*)/2i$ be nonnegative.

The following result is a simple generalization of the corresponding matrix theorem of I. Bendixson.

THEOREM 4.1. *The spectrum of a bounded dissipative operator* $A \in \mathfrak{R}$ *lies in the closed halfplane* $\operatorname{Im}\lambda \geqq 0$. *Moreover,*

$$(4.1) \qquad |(A - \lambda I)^{-1}| \leqq 1/|\operatorname{Im}\lambda| \text{ for } \operatorname{Im}\lambda < 0.$$ [5]

PROOF. In fact, if $\lambda = \alpha - i\beta$, where $\beta > 0$, then

$$\operatorname{Im}[(Af,f) - \lambda(f,f)] = (A_{\mathscr{I}}f,f) + \beta(f,f) \geqq \beta(f,f),$$

and so for $|f| = 1$,

$$|(A - \lambda I)f| \geqq |((A - \lambda I)f,f)| \geqq \beta|f|^2 = \beta.$$

It follows that λ is not an eigenvalue of the operator A and moreover that the operator $A - \lambda I$ maps all of $\mathfrak{H}$ linearly and continuously onto some subspace $\mathfrak{H}_1 \subseteq \mathfrak{H}$. It remains to show that $\mathfrak{H}_1 = \mathfrak{H}$. Assuming the contrary, we can assert that there is a vector $h \neq 0$ orthogonal to $\mathfrak{H}_1 = (A - \lambda I)\mathfrak{H}$; but then $(A^* - \bar{\lambda}I)h = 0$ and consequently $-\bar{\lambda}$ will be an eigenvalue of the dissipative operator $-A^*$. Since $\operatorname{Im}(-\bar{\lambda}) < 0$, this (as was already proved) is impossible.

Thus for $\operatorname{Im}\lambda < 0$ a bounded resolvent $(A - \lambda I)^{-1}$ always exists. Hence it is obvious that the inequality $|(A - \lambda I)f| \geqq \beta|f|$ is equivalent to (4.1). The theorem is proved.

2. The bound (4.1) enables us to establish the following result.

LEMMA 4.1.[6] *Let A be a bounded dissipative operator (in particular, a selfadjoint operator), and let $B = A + T$, where $T \in \mathfrak{S}_1$. Then for any θ_0 $(0 < \theta_0 < \pi/2)$ the limit relation*

$$(4.2) \qquad \overline{\lim_{\rho \to \infty}} \left[\frac{1}{\rho} \ln |D_{B/A}(\rho e^{i\theta})| \right] \leqq 0$$

holds uniformly in the sector U_{θ_0}

$$(4.3) \qquad |\pi/2 - \theta| \leqq \theta_0.$$

PROOF. We have

$$D_{B/A}(\mu) = \det[(I - \mu B)(I - \mu A)^{-1}] = \det(I - \mu TS(\mu)),$$

where

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[5] This bound will be used in the more general Lemma V.6.1.

[6] This lemma will first be used in §7 (for the proof of Theorem 7.1).

$$S(\mu) = (I - \mu A)^{-1}.$$

Hence according to the bound (1.2)

$$|D_{B/A}(\mu)| \leqq \prod_{j=1}^{\infty} (1 + |\mu| \, s_j(TS(\mu))).$$

On the other hand, according to Theorem 4.1, for $\operatorname{Im} \mu > 0$ we have

$$|\mu S(\mu)| = |(A - \mu^{-1} I)^{-1}| \leqq \frac{1}{|\operatorname{Im} 1/\mu|} = \frac{|\mu|^2}{|\operatorname{Im} \mu|},$$

i.e.

$$|S(\mu)| \leqq \frac{1}{\sin\theta} \qquad \text{for} \quad \mu = \rho e^{i\theta} \qquad (0 < \theta < \pi),$$

and consequently

$$s_j(TS(\mu)) \leqq \frac{s_j(T)}{\sin\theta} \qquad (j = 1, 2, \cdots).$$

Thus

$$|D_{B/A}(\rho e^{i\theta})| \leqq \prod_{j=1}^{\infty} \left(1 + \frac{\rho s_j(T)}{\sin\theta}\right)$$

$$\leqq \prod_{j=1}^{n} \left(1 + \frac{\rho s_j(T)}{\sin\theta}\right) \exp\left\{ \frac{\rho}{\sin\theta} \sum_{j=n+1}^{\infty} s_j(T) \right\},$$

from which follows (4.2).

LEMMA 4.2. *Suppose that the conditions of Lemma* 4.1 *are fulfilled and suppose, moreover, that the operator B is also dissipative. Then the limit relation*

$$\lim_{\rho \to \infty} \left[\frac{1}{\rho} \ln |D_{B/A}(\rho e^{i\theta})| \right] = 0 \tag{4.4}$$

holds uniformly in the sector (4.3).

Proof. In fact, if both operators A and B are dissipative and $B - A \in \mathfrak{S}_1$, then according to Lemma 4.1, for any θ_0 $(0 < \theta_0 < \pi/2)$ the inequality (4.2), as well as the inequality obtained from it by interchanging the roles of the operators A and B, will hold uniformly in the sector (4.3). Since $\ln|D_{B/A}(\mu)| = -\ln|D_{A/B}(\mu)|$, it follows that the limit in (4.4) holds uniformly in the indicated sector.

These lemmas will be used in §§7—9.

§5. A theorem on the perturbation determinant of a dissipative operator

A bounded operator K is called a *contraction* if $|Kf| \leqq |f|$ for every $f \in \mathfrak{H}$. One has the following result:

THEOREM 5.1. *Let* $A = G + iH$ $(H = A_{\mathscr{J}})$ *be a bounded dissipative operator, and let* $B = G + iF$, *where*

$$-H \leqq F \leqq H. \tag{5.1}$$

Then for $\operatorname{Im} \lambda < 0$ *the operator*

$$W(\lambda) = I - i(H - F)^{1/2}(A - \lambda I)^{-1}(H - F)^{1/2} \tag{5.2}$$

is a contraction, i.e.,

$$|W(\lambda) f| \leqq |f| \qquad (f \in \mathfrak{H}). \tag{5.3}$$

PROOF. Let us introduce, for conciseness, the notation $T = H - F$, $R(\lambda) = (A - \lambda I)^{-1}$; then

$$W(\lambda) = I - iT^{1/2}R(\lambda)\, T^{1/2},$$

and so

$$\begin{aligned} &I - W^*(\lambda)\, W(\lambda) \\ &\quad = iT^{1/2}R(\lambda)\, T^{1/2} - iT^{1/2}R^*(\lambda)\, T^{1/2} - T^{1/2}R^*(\lambda)\, TR(\lambda)\, T^{1/2}. \end{aligned} \tag{5.4}$$

On the other hand, since $R^*(\lambda) = (A^* - \bar{\lambda} I)^{-1}$,

$$\begin{aligned} R(\lambda) - R^*(\lambda) &= R^*(\lambda)\,((A^* - \lambda I) - (A - \lambda I))\, R(\lambda) \\ &= -2iR^*(\lambda)\, HR(\lambda) + 2i \operatorname{Im} \lambda R^*(\lambda)\, R(\lambda). \end{aligned}$$

Multiplying the first and third terms of this equality from the right and left by $T^{1/2}$, and then by i, we obtain, after comparison with (5.4),

$$\begin{aligned} &I - W^*(\lambda)\, W(\lambda) \\ &\qquad = T^{1/2}[R^*(\lambda)(H + F)\, R(\lambda) - 2\operatorname{Im} \lambda R^*(\lambda)\, R(\lambda)\,]T^{1/2}. \end{aligned} \tag{5.5}$$

It follows that

$$I - W^*(\lambda)\, W(\lambda) \geqq 0 \qquad \text{for} \qquad \operatorname{Im} \lambda < 0,$$

which is equivalent to the inequality (5.3). The theorem is proved.

THEOREM 5.2. *If the operators* A *and* B *satisfy the conditions of the preceding theorem and* $\operatorname{sp} H < \infty$, *then*

$$|D_{B/A}(\mu)| \leqq 1 \qquad \textit{for} \qquad \operatorname{Im} \mu > 0. \tag{5.6}$$

PROOF. Since, together with $H (\in \mathfrak{S}_1)$, the nonnegative operator $T = H - F$ $(\leqq 2H)$ belongs to $\mathfrak{S}_1$, the determinant

$$D_{B/A}(\mu) = \det(I + i\mu T(I - \mu A)^{-1})$$

has meaning. According to result 5 of §1 we can also write

$$D_{B/A}(\mu) = \det(I + i\mu T^{1/2}(I - \mu A)^{-1}T^{1/2}) = \det W(1/\mu).$$

Consequently

$$D_{B/A}(\mu) = \prod_{j=1}^{\infty}(1 + \epsilon_j(\mu)), \tag{5.7}$$

where $\epsilon_j(\mu)$ $(j = 1, 2, \cdots)$ is the complete system of eigenvalues of the operator

$$C(\mu) = i\mu T^{1/2}(I - \mu A)^{-1}T^{1/2}.$$

On the other hand, since $1 + \epsilon_j(\mu)$ $(j = 1, 2, \cdots)$ are the eigenvalues of the operator $W(1/\mu) = I + C(\mu)$, whose norm, by Theorem 5.1, does not exceed unity for $\operatorname{Im}\mu > 0$, we have

$$|1 + \epsilon_j(\mu)| \leqq 1 \qquad \text{for} \qquad \operatorname{Im}\mu > 0 \qquad (j = 1, 2, \cdots). \tag{5.8}$$

Thus inequality (5.6) is a consequence of the relations (5.7) and (5.8). The theorem is proved.

REMARK 5.1. In essence, the condition that $\operatorname{sp} H$ be finite was not used for the proof of Theorem 5.2, but only for the condition $\operatorname{sp}(H - F) < \infty$ which it implies, which was necessary in order that the infinite perturbation determinant $D_{B/A}(\mu)$ have meaning.

It is easily seen that Theorem 5.2 can be obtained as an immediate corollary of a result in which the parameter μ does not appear.

Let $A = G + iH$ $(G = A_{\mathscr{R}})$ be a bounded dissipative operator which has an inverse, and let $B = G + iF$, where $-H \leqq F \leqq H$ and $\operatorname{sp}(H - F) < \infty$. *Then*

$$|\det(BA^{-1})| \leqq 1. \tag{5.9}$$

This result is, possibly, new even for the case in which A and B are operators acting in a finite-dimensional space (i.e. for the case in which it can be formulated in terms of the theory of finite-dimensional matrices).

This result can be obtained as a corollary of the appropriate generalization of Theorem 5.1. In fact,

$$\det(BA^{-1}) = \det(I - i(H - F)A^{-1}). \tag{5.10}$$

On the other hand, as a generalization of Theorem 5.1 we can assert that the operator

$$W = I - i(H - F)^{1/2}A^{-1}(H - F)^{1/2}$$

is a contraction; the verification can be carried out similarly to that

of the corresponding result for the operator $W(\lambda)$.

Following this, it remains to note that according to (5.10) and (1.8), $\det(BA^{-1}) = \det W$, from which follows (5.9).

§6. The determinants $D_{A^*/A}(\lambda)$ and $D_{A_{\mathscr{R}}/A}(\lambda)$ for a dissipative operator A with nuclear imaginary component

1. Theorem 5.1 should be regarded as a generalization of a result of M. S. Livšic [2]. In fact, if we put $F = -H$ in (5.1), then we obtain the operator-function

$$(6.1) \qquad W(\lambda) = I - 2iH^{1/2}(A - \lambda I)^{-1}H^{1/2},$$

which he calls the characteristic operator-function of a dissipative operator, and Theorem 5.1 leads to the corresponding theorem of this author.[7)]

For $F = -H$, the relation (5.4) gives $W^*(\lambda)W(\lambda) = I$ for any real point λ which is regular for A. Similarly it can be proved that $W(\lambda)W^*(\lambda) = I$. Thus in this case we can supplement Theorem 5.1 by the assertion that the operator $W(\lambda)$ is unitary at any real point λ which is regular for A.

Noting that for $F = -H(\in \mathfrak{S}_1)$ we have $B = G - iH = A^*$ and

$$D_{B/A}(\lambda) = D_{A^*/A}(\lambda) = \det W(1/\lambda),$$

and also that for a completely continuous operator A the real characteristic numbers, and their multiplicities, of A and A^* are the same, we arrive at the basic part of the following result:

THEOREM 6.1. *Let $A = G + iH$ $(H = A_{\mathscr{I}})$ be a completely continuous dissipative operator, with $0 < \operatorname{sp} H < \infty$. Then*

$$(6.2) \qquad |D_{A^*/A}(\lambda)| \leqq 1 \ \textit{for} \ \operatorname{Im}\lambda \geqq 0,$$

and the equal sign holds only for real λ.

PROOF. After all that has been said, it remains to clarify why the equal sign in (6.2) cannot hold for $\operatorname{Im}\lambda > 0$. Assuming that the

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7) M. S. Livšic [2] defined the characteristic operator-function $W_A(\lambda)$ for any operator $A = G + iH(\in \mathfrak{R})$ by the formula

$$W_A(\lambda) = I + 2i \operatorname{sign} H H_a^{1/2}(A^* - \lambda I)^{-1}H_a^{1/2},$$

where H_a is the nonnegative square root of H^2 (the operator modulus of H). Therefore, to be precise, the function $W(\lambda)$, defined by (6.1) for the dissipative operator $A = G + iH$ $(H = H_a)$, should be regarded as the function $W_{A^*}(\lambda)$.

A more general definition of the characteristic function, and for a broader class of operators, can be found in papers by M. S. Brodskiĭ [1] and A. V. Straus [1].

equal sign in (6.2) is attained for some nonreal λ_0 ($\operatorname{Im}\lambda_0 > 0$), we can conclude on the basis of the maximum modulus principle for an analytic function that $D_{A^*/A}(\lambda) \equiv \text{const}$.

On the other hand, by the general formula (3.5) we have

$$\frac{d}{d\lambda}\ln D_{A^*/A}(\lambda) = \frac{1}{\lambda}\operatorname{sp}\{(I-\lambda A^*)^{-1} - (I-\lambda A)^{-1}\}$$
$$= \operatorname{sp}\{(I-\lambda A)^{-1}(A-A^*)(I-\lambda A^*)^{-1}\}$$
$$= 2i\operatorname{sp}\{(I-\lambda A)^{-1}H(I-\lambda A^*)^{-1}\}.$$

Consequently, for λ with small modulus,

$$\text{(6.3)}\qquad \frac{d}{d\lambda}\ln D_{A^*/A}(\lambda) = 2i\operatorname{sp}H + O(\lambda) \qquad (\lambda\to 0).$$

We have arrived at a contradiction, since by hypothesis $\operatorname{sp}H > 0$. The theorem is proved.

REMARK 6.1. The equality

$$\text{(6.4)}\qquad |D_{A^*/A}(\lambda)| = 1 \qquad (-\infty < \lambda < \infty)$$

is valid for any (not only dissipative) completely continuous operator with imaginary component $A_{\mathcal{J}} \in \mathfrak{S}_1$.

In fact, we have

$$D_{A^*/A}(\lambda) = D_{A^*/G}(\lambda)\, D_{G/A}(\lambda) = \frac{D_{G/A}(\lambda)}{D_{G/A^*}(\lambda)}.$$

On the other hand, as is not hard to see,[8]

$$D_{A^*/G}(\lambda) = \bar{D}_{A/G}(\lambda),$$

i.e. for any complex λ

$$D_{A^*/G}(\bar{\lambda}) = \overline{D_{A/G}(\lambda)}.$$

Thus

$$\text{(6.5)}\qquad D_{A^*/A}(\lambda) = \frac{D_{G/A}(\lambda)}{\bar{D}_{G/A}(\lambda)},$$

from which follows (6.4).

The relation (6.5) will play an important role in the sequel.

2. On the basis of Theorem 6.1 it is easy to prove the following theorem.

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[8] We recall that $\bar{f}(\lambda)$ denotes the function $\overline{f(\bar{\lambda})}$.

THEOREM 6.2. *The determinant* $D_{A^*/A}(\lambda)$ *is a meromorphic function which admits the expansion*

$$D_{A^*/A}(\lambda) = e^{2ia\lambda}\prod_j \frac{1-\lambda/\bar{\mu}_j}{1-\lambda/\mu_j}, \tag{6.6}$$

where $\{\mu_j\}$ *is the complete system of nonreal characteristic numbers of the dissipative operator* A, *and*

$$a = \operatorname{sp} H - \sum_j \operatorname{Im}\lambda_j \qquad (\lambda_j = 1/\mu_j). \tag{6.7}$$

PROOF. By the preceding theorem, the meromorphic function $f(\lambda) = D_{A^*/A}(\lambda)$ has the properties

$$|f(\lambda)| = 1 \quad \text{for} \quad \operatorname{Im}\lambda = 0, \qquad |f(\lambda)| < 1 \quad \text{for} \quad \operatorname{Im}\lambda > 0.$$

According to an elementary theorem of one of the authors (cf. Ahiezer and Kreĭn [1] and Levin [1]) every meromorphic function $f(\lambda)$ having these properties admits the representation

$$f(\lambda) = e^{2ia\lambda}\prod_{j=1}^{\infty}\frac{1-\lambda/\bar{\mu}_j}{1-\lambda/\mu_j},$$

where $a \geqq 0$ and $\{\mu_j\}$ is the complete sequence of poles of the function $f(\lambda)$. For $f(\lambda) = D_{A^*/A}(\lambda)$ this sequence coincides with the complete sequence of nonreal characteristic numbers of the operator A. Thus we have obtained the representation (6.6) for $D_{A^*/A}(\lambda)$, and it remains only to find the value of the constant a.

According to (6.6), the power series expansion of $D_{A^*/A}(\lambda)$ in a neighborhood of the point $\lambda = 0$ begins with the terms

$$D_{A^*/A}(\lambda) = 1 + 2i\left[a + \sum_j \operatorname{Im}\lambda_j\right]\lambda + \cdots.$$

On the other hand, it follows from (6.3) that

$$D_{A^*/A}(\lambda) = 1 + 2i\lambda \operatorname{sp} H + O(\lambda^2).$$

A comparison of these expansions yields the formula (6.7).

3. We do not have to add much to obtain the following result.

THEOREM 6.3. *Let* $A = G + iH$ *be a completely continuous dissipative operator for which* $\operatorname{sp} H < \infty$. *Then*

$$1)\quad |D_{G/A}(\lambda)| \leqq 1 \quad \textit{for } \operatorname{Im}\lambda \geqq 0, \tag{6.8}$$

$$2)\quad D_{G/A}(\lambda) = \bar{D}_{G/A}(\lambda)\, e^{2ai\lambda}\prod_{j=1}^{\infty}\frac{1-\lambda/\bar{\mu}_j}{1-\lambda/\mu_j}, \tag{6.9}$$

where $\{\mu_j\}$ *is the complete system of nonreal characteristic numbers of the operator* A, *and the constant* a *has the value* (6.7).

PROOF. All the conditions of Theorem 5.2 are fulfilled for the operators A and $B = G$, and the inequality (5.6), applied to the operators A and G, yields (6.8). The relation (6.9) is an immediate consequence of (6.5) and (6.6). The theorem is proved.

§7. Dissipative Volterra operators with nuclear imaginary component

1. THEOREM 7.1. *Let* $A = G + iH$ *be a dissipative Volterra operator with* $\operatorname{sp} H < \infty$. *Then* $G \in \mathfrak{S}_p$ *for any* $p > 1$, *and the entire function*

$$\widetilde{D}_G(\lambda) = \prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{\alpha_j}\right) e^{\lambda/\alpha_j},$$

where $\{\alpha_j\}$ *is the complete system of characteristic numbers of the operator* G, *has the property that*

$$|\widetilde{D}_G(\lambda)| \leqq e^{|\operatorname{Im}\lambda| \operatorname{sp} H} \tag{7.1}$$

for any complex λ. *Moreover, the limit relation*

$$\lim_{\rho\to\infty}\left[\frac{1}{\rho}\ln|\widetilde{D}_G(\rho e^{i\theta})|\right] = |\sin\theta| \operatorname{sp} H \tag{7.2}$$

holds uniformly in any sector U_ϑ $(0 < \vartheta < \pi/2)$

$$|\pi/2 - \theta| \leqq \vartheta.$$

PROOF. In fact, under the conditions of the theorem the function $F(\lambda) = D_{G/A}(\lambda)$ will be entire with the complete system of zeros $\{\alpha_j\}$, and by virtue of the relation (6.8) we have

$$|F(\lambda)| \leqq 1 \quad \text{for } \operatorname{Im}\lambda \geqq 0, \tag{7.3}$$

and by (6.9), for any λ

$$\bar{F}(\lambda) = e^{-2i\lambda a} F(\lambda) \qquad (a = \operatorname{sp} H). \tag{7.4}$$

From (7.3) and (7.4) we deduce that

$$|F(\lambda)| = |\bar{F}(\bar{\lambda})| = e^{2a|\operatorname{Im}\lambda|}|F(\bar{\lambda})| \leqq e^{2a|\operatorname{Im}\lambda|} \quad \text{for } \operatorname{Im}\lambda \leqq 0. \tag{7.5}$$

Thus the entire function $F(\lambda)$ is of not greater than exponential type. Therefore the infinite sum $\sum_j |\alpha_j|^{-p}$ converges for any $p > 1$, which shows that $G \in \mathfrak{S}_p$. In particular, $G \in \mathfrak{S}_2$ and $A \in \mathfrak{S}_2$, and so the determinants $\widetilde{D}_G(\lambda)$ and $\widetilde{D}_A(\lambda)$ have meaning, and the latter equals unity identically (since A is a Volterra operator). According to (3.3) we have

$$(7.6) \qquad \widetilde{D}_G(\lambda) = \frac{\widetilde{D}_G(\lambda)}{\widetilde{D}_A(\lambda)} = e^{i\lambda \operatorname{sp} H} D_{G/A}(\lambda) = e^{i\lambda \operatorname{sp} H} F(\lambda).$$

Hence, recalling (7.3) and (7.5), we obtain the bound (7.1).

From this also follows the last assertion of the theorem, since according to Lemma 4.2 the limit relation

$$\lim_{\rho \to \infty} \left[\frac{1}{\rho} \ln | D_{G/A}(\rho e^{i\theta}) | \right] = 0$$

holds uniformly in any sector U_ϑ $(0 < \vartheta < \pi/2)$.

2. An important deduction can be made from the theorem just proved.

THEOREM 7.2. *Let $A = G + iH$ $(H = A_{\mathscr{I}})$ be an arbitrary dissipative Volterra operator with* $\operatorname{sp} H < \infty$. *Then*

$$(7.7) \qquad \lim_{r \to \infty} \frac{n_+(r; G)}{r} = \lim_{r \to \infty} \frac{n_-(r; G)}{r} = \frac{1}{\pi} \operatorname{sp} H$$

and, moreover,

$$(7.8) \qquad \int_0^r \frac{n(\rho; G)}{\rho} \, d\rho \leqq \frac{2}{\pi} r \operatorname{sp} H \qquad (0 < r < \infty).$$

Here $n_+(r; G)$ and $n_-(r; G)$ denote the number of characteristic numbers of the operator G in the closed intervals $[0, r]$ and $[-r, 0]$ respectively, and

$$n(r; G) = n_+(r; G) + n_-(r; G).$$

PROOF. In fact, the relations (7.7) follow at once from the bound (7.1) on the basis of a theorem of N. Levinson (cf. result B), §8.1).

To establish (7.8) we shall use a well-known theorem of Jensen (see Levin [1], Chapter I, §5), according to which

$$(7.9) \qquad \int_0^r \frac{n(\rho; G)}{\rho} \, d\rho = \frac{1}{2\pi} \int_0^{2\pi} \ln | \widetilde{D}_G(re^{i\theta}) | \, d\theta.$$

By (6.8) and (7.6), the integral on the right side of this equality does not exceed

$$\frac{r}{2\pi} \operatorname{sp} H \int_0^{2\pi} |\sin \theta| \, d\theta = \frac{2r}{\pi} \operatorname{sp} H,$$

from which follows the inequality (7.8).

We remark that, by virtue of (7.7),

$$\lim_{r \to \infty} \frac{1}{r} \int_0^r \frac{n(\rho; G)}{\rho} \, d\rho = \frac{2}{\pi} \operatorname{sp} H.$$

Let us note two corollaries of the theorem.

COROLLARY 7.1. *Under the conditions of the theorem,*

$$\frac{n(r;G)}{r} \leqq c \operatorname{sp} H,$$

where for the constant c one can take $2e/\pi$.

In fact, since $n(r;G)$ is a nondecreasing function, we have

$$n(r;G) \leqq \int_r^{er} n(\rho;G)\frac{d\rho}{\rho} \leqq \int_0^{er} \frac{n(\rho;G)\,d\rho}{\rho} \leqq \frac{2e}{\pi} r \operatorname{sp} H.$$

COROLLARY 7.2. *Under the conditions of the theorem,*

$$\lim_{n\to\infty} ns_n(A) = \frac{2}{\pi}\operatorname{sp} H. \tag{7.10}$$

In fact, it follows from (7.7) that

$$\lim_{r\to\infty}\frac{n(r;G)}{r} = \lim_{r\to\infty}\frac{n_+(r;G)+n_-(r;G)}{r} = \frac{2}{\pi}\operatorname{sp} H.$$

This relation also shows that

$$\lim_{n\to\infty} ns_n(G) = \frac{2}{\pi}\operatorname{sp} H. \tag{7.11}$$

On the other hand,

$$\lim_{n\to\infty} ns_n(H) = 0, \tag{7.12}$$

since $\operatorname{sp} H < \infty$.

Recalling Theorem II.2.3 of K. Fan, we obtain from (7.11) and (7.12) the relation (7.10) for the operator $A = G + iH$, the sum of the operators G and iH.

3. We shall show that for a finite-dimensional H the integral appearing in (7.8) admits a simple lower bound.

Indeed, if N is the dimension of $\mathfrak{R}(H)$, then

$$|D_{A/G}(\lambda)| = |\det(1 + i\lambda H(I-\lambda G)^{-1})| \leqq \prod_{j=1}^{N}(1+\sigma_j(\lambda)),$$

where $\sigma_j(\lambda)$ $(j = 1,2,\cdots,N)$ are the s-numbers of the operator $C(\lambda) = i\lambda H(I-\lambda G)^{-1}$. Since G is a selfadjoint operator,

$$|\lambda(I-\lambda G)^{-1}| = \left|\left(\frac{1}{\lambda}I - G\right)^{-1}\right| \leqq \frac{1}{|\operatorname{Im}(1/\lambda)|} = \frac{|\lambda|}{|\sin\theta|} \qquad (\lambda = re^{i\theta}),$$

and consequently

$$\sigma_j(\lambda) = s_j(C(\lambda)) \leqq \frac{|\lambda|}{|\sin\theta|}\, s_j(H) \qquad (j = 1, 2, \cdots, N).$$

Thus

$$|D_{A/G}(\lambda)| \leqq \prod_{j=1}^{N}\left(1 + \frac{rs_j(H)}{|\sin\theta|}\right) \qquad (\lambda = re^{i\theta}).$$

Hence for $\operatorname{Im}\lambda \neq 0$

$$|\widetilde{D}_G(\lambda)| = |e^{\,i\lambda\,\mathrm{sp}\,H} D_{A/G}^{-1}(\lambda)| \geqq e^{|\operatorname{Im}\lambda|\,\mathrm{sp}\,H}\prod_{j=1}^{N}\left(1 + \frac{rs_j(H)}{|\sin\theta|}\right)^{-1}$$

and consequently

$$\text{(7.13)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \ln|\widetilde{D}_G(re^{i\theta})|\, d\theta \geqq \frac{2}{\pi}\, r\,\mathrm{sp}\,H - \sum_{j=1}^{N}\Phi(rs_j(H)),$$

where

$$\begin{aligned}\Phi(r) &= \frac{1}{2\pi}\int_0^{2\pi} \ln\left(1 + \frac{r}{|\sin\theta|}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \ln(|\sin\theta| + r)\, d\theta + \ln 2.\end{aligned}$$

Obviously, for $r \to \infty$

$$\begin{aligned}\Phi(r) &= \ln 2 + \ln r + \frac{1}{2\pi}\int_0^{2\pi} \ln\left(1 + \frac{|\sin\theta|}{r}\right) d\theta \\ &= \ln 2 + \ln r - \frac{4}{r} + O\left(\frac{1}{r^2}\right).\end{aligned}$$

Thus a comparison of (7.9) and (7.13), taking into account the preceding asymptotic expression, yields

$$\frac{2}{\pi}\,\mathrm{sp}\,H \geqq \frac{1}{r}\int_0^r \frac{n(\rho; G)}{\rho}\, d\rho \geqq \frac{2}{\pi}\,\mathrm{sp}\,H - \frac{N\ln r}{r} - \sum_{j=1}^{N}\frac{\ln(2s_j(H))}{r} + O\left(\frac{1}{r^2}\right).$$

4. The simplest example of a dissipative Volterra operator. The simple[9] dissipative Volterra operators A with $A_{\mathscr{J}} \in \mathfrak{S}_1$ can be divided into classes by grouping together those operators whose imaginary components have the same dimension.

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[9] A dissipative Volterra operator is said to be *simple*, if it does not vanish on any vector $f \neq 0$. (The general concept of a simple dissipative operator will be used systematically, starting with §4 of Chapter V.)

The simplest of these will be the class of simple dissipative Volterra operators with a one-dimensional imaginary component. A model example is the integral operator J in $L_2(0,1)$, defined by

$$(Jf)(t) = 2i\int_0^t f(s)\,ds.$$

Indeed, the adjoint J^* acts according to the formula

$$(J^*f)(t) = -2i\int_t^1 f(s)\,ds.$$

Therefore

$$(J_{\mathscr{I}}f)(t) = \int_0^1 f(s)\,ds, \tag{7.14}$$

$$(J_{\mathscr{R}}f)(t) = i\int_0^1 \operatorname{sign}(t-s)\,f(s)\,ds. \tag{7.15}$$

From (7.14) it is clear that $J_{\mathscr{I}}$ is a one-dimensional operator with eigenvalue equal to unity, and eigenfunction $e(t) \equiv 1$.

Let us calculate the spectrum of eigenvalues of the operator $J_{\mathscr{R}}$. If $\phi(t) = \lambda(J_{\mathscr{R}}\phi)(t)$, then, according to (7.15), this means that

$$\phi(t) = i\lambda\int_0^1 \operatorname{sign}(t-s)\,\phi(s)\,ds.$$

It is easily seen that this integral equation is equivalent to the boundary value problem

$$\phi'(t) - 2i\lambda\phi(t) = 0, \qquad \phi(0) + \phi(1) = 0.$$

The functions

$$\phi_n(t) = e^{-(2n+1)\pi i t} \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

form a complete system of eigenfunctions of this boundary value problem. The corresponding characteristic numbers are

$$\alpha_n = (2n+1)\pi/2 \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Since $\operatorname{sp} J_{\mathscr{I}} = 1$, the values obtained for the α_n $(n = 0, \pm 1, \pm 2, \cdots)$ are in agreement with the asymptotic formulas (7.7).

It turns out that one has the following remarkable theorem of M. S. Livšic [1] (see also Brodskiĭ [2], Gohberg and Kreĭn [1]).

Every simple Volterra operator A with a one-dimensional imaginary component is unitarily equivalent to the operator cJ, where $c = \operatorname{sp} A_{\mathscr{I}}$.

5. An application to integral operators. Let $L_{2,\sigma}^{(r)}(a,b)$ be the Hilbert space of vector-functions $f(t) = \{f_j(t)\}_1^r$, generated on the interval $[a,b]$

by a matrix-function $\sigma(t)$ having the properties a), b) from §10.5, Chapter III, and let $\mathscr{K}(t,s) = \| k_{pq}(t,s) \|_1^r$ $(a \leq t, s \leq b)$ be a Hermitian nonnegative continuous matrix kernel.

1. *For the distribution functions* $n_\pm(r; G)$ *of the characteristic numbers of the operator* G, *defined in* $L_{2,\sigma}^{(r)}(a,b)$ *by*

$$(Gf)(t) = i \int_a^b \operatorname{sign}(t-s) \mathscr{K}(t,s)\, d\sigma(s)\, f(s),$$

we have the relation

$$\lim_{r \to \infty} \frac{n_\pm(r; G)}{r} = \frac{1}{\pi} \int_a^b \mathrm{sp}\, [\mathscr{K}(s,s)\, d\sigma(s)\,]. \tag{7.16}$$

This result can be obtained as a simple corollary of Theorem 7.2. In fact, the operator G is the real component of the dissipative Volterra operator

$$(Af)(t) = 2i \int_t^b \mathscr{K}(t,s)\, d\sigma(s)\, f(s).$$

Moreover, according to §10.5, Chapter III, the trace of the operator $A_{\mathscr{J}}$ can be calculated from the formula

$$\mathrm{sp}\, A_{\mathscr{J}} = \int_a^b \mathrm{sp}\, [\mathscr{K}(s,s)\, d\sigma(s)\,].$$

§8. Nondissipative operators with nuclear imaginary component

1. We will have to draw upon some new methods of the theory of functions.

We denote by $(\mathfrak{A}_+)$ the class of functions $F(\lambda)$ which are holomorphic in the halfplane $\operatorname{Im}\lambda > 0$ and can be represented there as the quotient of two bounded holomorphic functions. By a well-known theorem of R. Nevanlinna (see I. I. Privalov [1], Chapter II, §§1, 2), a function $F(\lambda)$, holomorphic for $\operatorname{Im}\lambda > 0$, belongs to the class $(\mathfrak{A}_+)$ if and only if there exists a nonnegative harmonic function $u(\lambda)$ $(\operatorname{Im}\lambda > 0)$ such that $\ln|F(\lambda)| \leq u(\lambda)$.

From the definition of the class $(\mathfrak{A}_+)$ it is comparatively simple to obtain the following two properties.

1. *The class* $(\mathfrak{A}_+)$ *is an algebra, i.e. if* $F_1, F_2 \in (\mathfrak{A}_+)$, *then the product* $F_1F_2 \in (\mathfrak{A}_+)$, *and for any complex* c_1, c_2 *the linear combination* $c_1F_1 + c_2F_2 \in (\mathfrak{A}_+)$.

2. *If* $F_1, F_2 \in (\mathfrak{A}_+)$, *and* F_1/F_2 *is a holomorphic function for* $\operatorname{Im}\lambda > 0$, *then* $F_1/F_2 \in (\mathfrak{A}_+)$.

We define the class $(\mathfrak{A}_-)$ of functions $F(\lambda)$, holomorphic in the halfplane $\operatorname{Im}\lambda < 0$, analogously to $(\mathfrak{A}_+)$. Obviously, if $F(\lambda) \in (\mathfrak{A}_+)$, then

$F(-\lambda) \in (\mathfrak{A}\ \ )$ and $F(\lambda) \in (\mathfrak{A}\ \ )$.

The following result (M. G. Kreĭn [3]) will play an essential role in the sequel.

A) *If the entire function* $f(\lambda)$ *belongs to the class* $(\mathfrak{A}_+)$ *in the upper halfplane, and to the class* $(\mathfrak{A}_-)$ *in the lower halfplane, then it is of not greater than exponential type, i.e.*

$$\ln|f(\lambda)| = O(|\lambda|) \quad \text{for} \quad |\lambda| \to \infty \tag{8.1}$$

and

$$\int_{-\infty}^{\infty} \frac{|\ln|f(x)||}{1+x^2}\, dx < \infty. \tag{8.2}$$

We shall supplement this result by the following:

B) *If the real entire function* $f(\lambda)$ $(f(0) = 1)$ *with only real zeros* $\{\alpha_j\}$ *satisfies the conditions* (8.1) *and* (8.2), *then*

$$f(\lambda) = \lim_{r\to\infty} \prod_{|\alpha_j| \le r} \left(1 - \frac{\lambda}{\alpha_j}\right) \tag{8.3}$$

and

$$\lim_{r\to\infty} \frac{n_+(r;f)}{r} = \lim_{r\to\infty} \frac{n_-(r;f)}{r} = \frac{1}{\pi} \lim_{r\to\infty} \frac{\ln|f(ir)|}{r} \quad (<\infty). \tag{8.4}$$

Here $n_+(r;f)$ and $n_-(r;f)$ denote the number of zeros α_j contained in the closed intervals $[0,r]$ and $[-r,0]$, respectively.

The equality (8.4) was established by N. Levinson (see Levin [1], Chapter II, §4). The relation (8.3) is a corollary of a more general theorem of Levin [1] (Chapter V, §4, Theorem 11).

2. We begin with the following result.

LEMMA 8.1. *Let* $A = G + iH$ *be a completely continuous operator with component* $A_{\mathscr{J}} = H \in \mathfrak{S}_1$, *and let* $\{\mu_j^+\}$ *and* $\{\mu_j^-\}$ *be the complete sequences of the characteristic numbers of the operator* A *in the upper and lower halfplanes respectively. Then in the upper halfplane*

$$\prod_j \frac{1-\lambda/\mu_j^+}{1-\lambda/\bar{\mu}_j^+}\, D_{G/A}(\lambda) \in (\mathfrak{A}_+), \tag{8.5}$$

and in the lower halfplane

$$\prod_j \frac{1-\lambda/\mu_j^-}{1-\lambda/\bar{\mu}_j^-}\, D_{G/A}(\lambda) \in (\mathfrak{A}_-). \tag{8.6}$$

PROOF. The operator H can be represented as the difference of nonnegative orthogonal operators:

$$H = H_+ - H_- .$$

We put

$$(8.7) \qquad H_1 = H_+ + H_-, \qquad A_1 = G + iH_1$$

and consider the decomposition

$$(8.8) \qquad D_{G/A}(\lambda) = \frac{D_{G/A_1}(\lambda)}{D_{A/A_1}(\lambda)} .$$

Since $-H_1 \leqq H \leqq H_1$, by Theorem 5.2 we have

$$(8.9) \qquad |D_{A/A_1}(\lambda)| \leqq 1 \qquad \text{for} \quad \operatorname{Im}\lambda \geqq 0.$$

By the same theorem (or Theorem 6.3),

$$|D_{G/A_1}(\lambda)| \leqq 1 \qquad \text{for} \quad \operatorname{Im}\lambda \geqq 0.$$

The only zeros of the function $D_{A/A_1}(\lambda)$ in the halfplane $\operatorname{Im}\lambda > 0$ are the characteristic numbers μ_j^+ $(j = 1, 2, \cdots)$.

From (8.9) follows

$$(8.10) \qquad D_{A/A_1}(\lambda) = \prod_j \frac{1 - \lambda/\mu_j^+}{1 - \lambda/\bar{\mu}_j^+} f(\lambda),$$

where $f(\lambda)$ is some function which is holomorphic and different from zero for $\operatorname{Im}\lambda > 0$, and satisfies the condition

$$|f(\lambda)| \leqq 1.$$

According to (8.8) and (8.10),

$$(8.11) \qquad \prod_j \frac{1 - \lambda/\mu_j^+}{1 - \lambda/\bar{\mu}_j^+} D_{G/A}(\lambda) = \frac{D_{G/A_1}(\lambda)}{f(\lambda)},$$

and since the right side is the quotient of two analytic functions which are bounded for $\operatorname{Im}\lambda > 0$, (8.5) is proved. The proof of (8.6) is similar. The lemma is proved.

3. We can now obtain without difficulty a generalization of Theorem 6.2 to the case of nondissipative operators.

THEOREM 8.1. *Let $A = G + iH$ be a completely continuous operator with component $A_{\mathcal{J}} = H \in \mathfrak{S}_1$. Then*

$$(8.12) \qquad D_{A^*/A}(\lambda) = e^{2ia\lambda} \prod_j \frac{1 - \lambda/\bar{\mu}_j}{1 - \lambda/\mu_j},$$

where $\{\mu_j\}$ is the complete sequence of all nonreal characteristic numbers of the operator A, and

$$a = \operatorname{sp} H - \sum_j \operatorname{Im} \frac{1}{\mu_j}. \tag{8.13}$$

PROOF. Interchanging the roles of A and A^*, we can obtain for A^* a relation analogous to (8.11), namely

$$\prod_j \frac{1 - \lambda/\bar{\mu}_j}{1 - \lambda/\mu_j} D_{G/A^*}(\lambda) = D_{G/A_1}/f^*(\lambda), \tag{8.14}$$

where $f^*(\lambda)$, like $f(\lambda)$, is a function which is holomorphic and different from zero for $\operatorname{Im}\lambda > 0$, with modulus not exceeding unity.

Dividing (8.11) into (8.14), we find that

$$D_{A^*/A}(\lambda) = \prod_j \frac{1 - \lambda/\bar{\mu}_j}{1 - \lambda/\mu_j} \frac{f(\lambda)}{f^*(\lambda)}. \tag{8.15}$$

Since, according to (6.4),

$$|D_{A^*/A}(\lambda)| = 1 \qquad \text{for} \quad \operatorname{Im}\lambda = 0,$$

it follows from (8.15) that

$$|f(\lambda)| = |f^*(\lambda)| \qquad \text{for} \quad -\infty < \lambda < \infty.$$

Since the harmonic function $\ln|f(\lambda)|$ is nonpositive for $\operatorname{Im}\lambda \geqq 0$, it admits the representation (cf., in this regard, §9.1)

$$\ln|f(\lambda)| = \left(\gamma + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{p(x)\,dx}{|\lambda - x|^2}\right) \operatorname{Im}\lambda \qquad (\operatorname{Im}\lambda > 0), \tag{8.16}$$

where

$$p(x) = \ln|f(x)| = \ln|f^*(x)| \qquad (-\infty < x < \infty),$$

$$\gamma = \lim_{\rho\to\infty}\left[\frac{1}{\rho}\ln|f(i\rho)|\right] \leqq 0.$$

We can write down a representation for $\ln|f^*(\lambda)|$ similar to (8.16), with the same $p(x)$, but with some other constant γ^*. Hence

$$\ln|f(\lambda)| = \ln|f^*(\lambda)| - 2a\operatorname{Im}\lambda, \quad \left|e^{-2ia\lambda}\frac{f(\lambda)}{f^*(\lambda)}\right| = 1 \qquad (\operatorname{Im}\lambda \geqq 0),$$

where $2a = \gamma^* - \gamma$. Since $f(0)/f^*(0) = 1$, we conclude that

$$\frac{f(\lambda)}{f^*(\lambda)} = e^{2ia\lambda},$$

which together with (8.15) yields (8.12) for $\operatorname{Im}\lambda \geqq 0$ (with $a = (\gamma^* - \gamma)/2$).

Since the left and right sides of the equality (8.12) are meromorphic

functions, this equality is valid over the entire complex plane. It remains to verify the equality (8.13). Just as in the proof of Theorem 6.2, it can be verified by comparing the expansions in powers of λ, in a neighborhood of the origin, of its right and left sides. The theorem is proved.

4. We shall now formulate a result which generalizes and supplements the most essential portion of Theorem 7.2.

THEOREM 8.2. *Let $A = G + iH$ be a Volterra operator with component $A_{\mathscr{J}} = H \in \mathfrak{S}_1$. Then the determinant $\widetilde{D}_G(\lambda)$ is an entire function satisfying the conditions* (8.1) *and* (8.2); *its zeros, i.e. the characteristic numbers α_j $(j = 1, 2, \cdots)$ of the operator G, are such that*

$$(8.17) \qquad 1) \qquad \lim_{r \to \infty} \sum_{|\alpha_j| \leq r} \frac{1}{\alpha_j} = 0;$$

$$(8.18) \qquad 2) \qquad \lim_{r \to \infty} \frac{n_+(r; G)}{r} = \lim_{r \to \infty} \frac{n_-(r; G)}{r} = \frac{h}{\pi},$$

where

$$(8.19) \qquad |\operatorname{sp} H| \leq h \leq \operatorname{sp}|H| \ (= |H|_1),$$

and 3) *there exists a constant γ $(\leq 4e)$, not depending upon the operator A, such that*

$$\frac{n(r; G)}{r} \leq \frac{\gamma}{\pi} \operatorname{sp}|H|.$$

All the assertions of this theorem, with the exception of the bound $h \leq |H|_1$ and the assertion 3), will be proved on the basis of what has been said earlier. To obtain the indicated bound, we have to use the following result, which is of interest in itself.

C) *Let $A = G + iH$ be a Volterra operator with component $A_{\mathscr{J}} = H \in \mathfrak{S}_1$. Then $G \in \mathfrak{S}_\Omega$,*[10] *and moreover*

$$\left.\begin{matrix} \displaystyle\sum_{j=1}^{n} \lambda_j^+(G) \\ \displaystyle\sum_{j=1}^{n} \lambda_j^-(G) \end{matrix}\right\} \leq \frac{2}{\pi} |H|_1 \sum_{j=1}^{n} \frac{1}{2j-1} \qquad (n = 1, 2, \cdots).$$

This result was established by the authors in their communication [**3**]; its proof is based on the theory of triangular representations of nonselfadjoint operators and in particular on the theorem of Livšic which was formulated at the end of §7.3.

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[10] For the definition of the s.n. ideal $\mathfrak{S}_\Omega$ see §15.5, Chapter III.

The theory of triangular representations also enables one to prove the following simple result (Gohberg and Kreĭn [**2, 3**]).[11]

D) *Let* $A = G + iH$ *be a Volterra operator with imaginary component* $A_{\mathscr{I}} = H \in \mathfrak{S}_p$ $(1 \leqq p < \infty)$. *Then the operator* A *admits an inessential extension* $\widetilde{A} = \widetilde{G} + i\widetilde{H}$, *representable as a difference* $\widetilde{A} = A_1 - A_2$, *where* $A_1 = G_1 + i\widetilde{H}_+$ *and* $A_2 = G_2 + i\widetilde{H}_-$ *are dissipative Volterra operators with orthogonal imaginary components* $\widetilde{H}_+$, $\widetilde{H}_-$.

Let us clarify that an operator $\widetilde{A}$, acting in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$, is said to be an *inessential extension* of the operator A, if $\widetilde{A}\phi = A\phi$ for $\phi \in \mathfrak{H}$ and $\widetilde{A}\phi = 0$ for $\phi \in \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$.

The result D) for $p = 1$ will be used in the proof of the assertion 3) of Theorem 8.2.

PROOF OF THEOREM 8.2. Since by hypothesis A is a Volterra operator, $\widetilde{D}_A(\lambda) = 1$ and according to formula (3.3)

$$\widetilde{D}_G(\lambda) = \frac{\widetilde{D}_G(\lambda)}{\widetilde{D}_A(\lambda)} = D_{G/A}(\lambda)\, e^{i\lambda\,\mathrm{sp}\,H}.$$

By Lemma 8.1 the entire function $D_{G/A}(\lambda)$ belongs to $(\mathfrak{A}_\pm)$ and consequently $\widetilde{D}_G(\lambda) \in (\mathfrak{A}_\pm)$. On the basis of result A), the function $\widetilde{D}_G(\lambda)$ has the properties (8.1) and (8.2); consequently result B) is applicable to it.

Therefore, according to formula (8.4) of N. Levinson, the equality (8.18) holds for

$$h = \lim_{r\to\infty}\left[\frac{1}{r}\ln|\widetilde{D}_G(ir)|\right].$$

Applying to the regularized determinant $\widetilde{D}_G(\lambda)$ the representation (8.3) yields

$$\widetilde{D}_G(\lambda) = \prod_j\left(1 - \frac{\lambda}{\alpha_j}\right)\exp\left[\frac{\lambda}{\alpha_j}\right] = \lim_{r\to\infty}\prod_{|\alpha_j|\leqq r}\left(1 - \frac{\lambda}{\alpha_j}\right),$$

from which we obtain (8.17).

To prove that

$$h = \lim_{r\to\infty}\left[\frac{1}{r}\ln|\widetilde{D}_G(ir)|\right] \geqq |\mathrm{sp}\,H|, \tag{8.20}$$

we obviously have to consider only the case in which $\mathrm{sp}\,H \neq 0$. For definiteness we take $\mathrm{sp}\,H > 0$.

[11] A result of V. I. Macaev [2] makes it possible to replace the s.n. ideal $\mathfrak{S}_p$ in this result by the s.n. ideal $\mathfrak{S}_\omega$ (see also Gohberg and Kreĭn [7], result 2 of §4, Chapter III).

We shall use the relation

$$\widetilde{D}_G(\lambda) = e^{-i\lambda \operatorname{sp} H} D_{G/A}(\lambda) = e^{-i\lambda \operatorname{sp} H} \frac{D_{G/A_1}(\lambda)}{D_{A/A_1}(\lambda)}, \tag{8.21}$$

where the dissipative operator A_1 is defined by (8.7). By Lemma 4.1

$$\overline{\lim_{r\to\infty}}\left[\frac{1}{r}\ln|D_{A/A_1}(ir)|\right] \leqq 0,$$

and according to Lemma 4.2

$$\lim_{r\to\infty}\left[\frac{1}{r}\ln|D_{G/A_1}(ir)|\right] = 0.$$

It follows from (8.21) that

$$\begin{aligned} h &= \lim_{r\to\infty}\left[\frac{1}{r}\ln|\widetilde{D}_G(ir)|\right] \\ &= \operatorname{sp} H - \overline{\lim_{r\to\infty}}\left[\frac{1}{r}\ln|D_{A/A_1}(ir)|\right] \geqq |\operatorname{sp} H|. \end{aligned}$$

Thus inequality (8.20) is proved.

According to result C), for the sequence of positive characteristic numbers $\alpha_j^+ = 1/\lambda_j^+(G)$ $(j = 1, 2, \cdots)$ we have

$$\sup_n \left(\sum_{j=1}^n \frac{1}{\alpha_j^+} \Big/ \sum_{j=1}^n \frac{1}{j}\right) \leqq \frac{1}{\pi}|H|_1. \tag{8.22}$$

Consequently, always

$$\overline{\lim_{n\to\infty}} \left(\sum_{j=1}^n \frac{1}{\alpha_j^+} \Big/ \sum_{j=1}^n \frac{1}{j}\right) \leqq \frac{1}{\pi}|H|_1.$$

On the other hand, if $h > 0$ in (8.18), then for any $\epsilon > 0$

$$\frac{1}{\alpha_k^+} > \frac{h-\epsilon}{\pi}\frac{1}{k} \qquad (k > N_\epsilon),$$

and consequently

$$\overline{\lim_{n\to\infty}}\left[\sum_{j=1}^n \frac{1}{\alpha_j^+} \Big/ \sum_{j=1}^n \frac{1}{j}\right] > \frac{h-\epsilon}{\pi}.$$

Thus $h - \epsilon \leqq |H|_1$, and the bound (8.19) is established.

It remains to justify the assertion 3). For the case of a dissipative

operator A it was established earlier, since, according to Corollary 7.1, in this case

$$\sup_{0<r<\infty} \frac{n(r;G)}{r} \leqq \frac{2e}{\pi} \operatorname{sp} H. \tag{8.23}$$

In the general case we have to use result D), according to which every Volterra operator A admits an inessential extension $\widetilde{A} = \widetilde{G} + i\widetilde{H}$, representable as a difference $\widetilde{A} = A_1 - A_2$, where $A_1 = G_1 + i\widetilde{H}_+$ and $A_2 = G_2 + i\widetilde{H}_-$ are dissipative Volterra operators, and $\widetilde{H}_+$ and $\widetilde{H}_-$ are orthogonal to each other.

Since under an inessential extension of the operator A the quantities $n(r; A_{\mathscr{R}})$ and $\operatorname{sp}|A_{\mathscr{J}}|$ related to it do not change their values, we may assume without loss of generality that the operator A itself admits the representation $A = A_1 - A_2$, where $A_1 = G_1 + iH_+$ and $A_2 = G_2 + iH_-$ are dissipative Volterra operators. For A_1 and A_2 we will have

$$\sup_{0<r<\infty} \frac{n(r;G_1)}{r} \leqq \frac{2e}{\pi} \operatorname{sp} H_+, \qquad \sup_{0<r<\infty} \frac{n(r;G_2)}{r} \leqq \frac{2e}{\pi} \operatorname{sp} H_-. \tag{8.24}$$

It is easily seen that for any selfadjoint $G \in \mathfrak{S}_\infty$

$$\sup_{0<r<\infty} \frac{n(r;G)}{r} = \sup_{1 \leqq n < \infty} n s_n(G),$$

and therefore (8.24) shows that

$$\sup_n n s_n(G_1) \leqq \frac{2e}{\pi} \operatorname{sp} H_+, \qquad \sup_n n s_n(G_2) \leqq \frac{2e}{\pi} \operatorname{sp} H_-. \tag{8.25}$$

Since, on the other hand, $G_1 + G_2 = G$, by Corollary II.2.2 we have $s_{n+m-1}(G) \leqq s_n(G_1) + s_m(G_2)$ $(n, m = 1, 2, \cdots)$. Hence for $m = n$ and $m = n + 1$ we obtain

$$(2n-1)\, s_{2n-1}(G) \leqq 2\,[n s_n(G_1) + n s_n(G_2)\,],$$

$$2n s_{2n}(G) \leqq 2\,[n s_n(G_1) + (n+1)\, s_{n+1}(G_2)\,],$$

and so

$$\sup_n n s_n(G) \leqq 2\left[\sup_n n s_n(G_1) + \sup_n n s_n(G_2)\right].$$

Taking (8.25) into account, we arrive at property 3) with the value $\gamma = 4e$. The theorem is proved.

We remark that in the case of a dissipative Volterra operator A it

follows from (8.19) that $h = \operatorname{sp} H$. This fact was already known to us from Theorem 7.2. One can show that in the general case h can assume any value from the interval (8.19).

REMARK 8.1. Relation (8.17) can also be written in the form

$$\lim_{\epsilon \downarrow 0} \sum_{|\lambda_j| \geq \epsilon} \lambda_j(G) = 0. \tag{8.26}$$

This can be regarded as the assertion that *the "principle value of the trace" of the real component of a Volterra operator is equal to zero whenever the imaginary component of the operator is nuclear.*

If the Volterra operator $A = G + iH$ is itself in $\mathfrak{S}_1$ then $G \in \mathfrak{S}_1$, and so the trace of G has meaning and its principle value coincides with it, so that $\operatorname{sp} G = 0$. Applying this result to the operator iA, we also obtain $\operatorname{sp} H = 0$ and, consequently, in this case $\operatorname{sp} A = 0$.

We have again arrived at the result of V. B. Lidskiĭ (cf. Theorem III. 8.4).

REMARK 8.2. From the relation (8.26) one can make the following deduction: *if the Volterra operator A is representable in the form $A = C + T$, where C is a nonnegative operator and $T \in \mathfrak{S}_1$, then $A \in \mathfrak{S}_1$ and* $\operatorname{sp} C \leqq |T_{\mathscr{R}}|_1 \quad (\leqq |T|_1)$.

In fact, since $A_{\mathscr{I}} = T_{\mathscr{I}} \in \mathfrak{S}_1$,

$$\lim_{\epsilon \downarrow 0} \sum_{|\lambda| \geq \epsilon} \lambda_j(A_{\mathscr{R}}) = 0. \tag{8.27}$$

On the other hand, if we use the decompositions $A_{\mathscr{R}} = A_{\mathscr{R}}^{+} - A_{\mathscr{R}}^{-}$ and $T_{\mathscr{R}} = T_{\mathscr{R}}^{+} - T_{\mathscr{R}}^{-}$ of the operators $A_{\mathscr{R}}$ and $T_{\mathscr{R}}$ into orthogonal nonnegative terms, then $A_{\mathscr{R}} = (T_{\mathscr{R}}^{+} + C) - T_{\mathscr{R}}$ and so, according to Lemma II.1.2, we will have

$$\operatorname{sp} A_{\mathscr{R}}^{-} \leqq \operatorname{sp} T_{\mathscr{R}}^{-} \leqq |T_{\mathscr{R}}|_1 \leqq |T|_1 < \infty.$$

But then, according to (8.27), $\operatorname{sp} A_{\mathscr{R}}^{+} = \operatorname{sp} A_{\mathscr{R}}^{-}$, $A_{\mathscr{R}} \in \mathfrak{S}_1$, $\operatorname{sp} A_{\mathscr{R}} = 0$ and $A = A_{\mathscr{R}} + iA_{\mathscr{I}} \in \mathfrak{S}_1$. Moreover,

$$\operatorname{sp} C \leqq \operatorname{sp}(T_{\mathscr{R}}^{+} + C) = \operatorname{sp} T_{\mathscr{R}}^{-} \leqq |T_{\mathscr{R}}|_1 \leqq |T|_1.$$

5. The number h appearing in (8.18) can be calculated from the formula

$$h = \int_{\mathfrak{P}} |dP\,H\,dP|_1, \tag{8.28}$$

where $\mathfrak{P}$ is a maximal eigenchain of the operator A. This result requires the use of new concepts and methods, and will be proved in the authors' book [7].

Let us clarify this last concept and the meaning of formula (8.28).

A set $\mathfrak{P} = \{P\}$ of orthoprojectors which is closed in the sense of strong convergence is called a *chain* if it is ordered in the natural way and contains the projectors 0 and I. A chain $\mathfrak{P}$ is said to be *maximal* if it is not a proper part of any other chain.

A chain $\mathfrak{P}$ is said to be an *eigenchain* of an operator $A \in \mathfrak{R}$, if for every orthoprojector $P \in \mathfrak{P}$ the subspace $P\mathfrak{H}$ is invariant with respect to A. From a theorem of J. von Neumann and N. Aronszajn (see Aronszajn and Smith [1]) on the existence of a proper invariant subspace for any operator $A \in \mathfrak{S}_\infty$ one can deduce the following important result (cf. L. A. Sahnovič [1]).

Every operator $A \in \mathfrak{S}_\infty$ has at least one eigenchain which is maximal.

The formula (8.28) can be most simply interpreted as being equivalent to the following equality:

$$h = \inf \sum_{j=1}^{n} |\Delta P_j H \Delta P_j|_1, \tag{8.29}$$

where the infimum is extended over the set of all partitions $0 = P_0 < P_1 < \cdots < P_n = I$ ($P_j \in \mathfrak{P}$) of the chain $\mathfrak{P}$, and $\Delta P_j = P_j - P_{j-1}$.

We recall that, according to Theorem III.8.7, if the partition $\{P_j''\}_0^m$ is a refinement of the partition $\{P_j'\}_0^n$, then

$$\sum_{j=1}^{m} |\Delta P_j'' H \Delta P_j''|_1 \leqq \sum_{j=1}^{n} |\Delta P_j' H \Delta P_j'|_1.$$

REMARK 8.3. *It follows from* (8.18) *that*

$$\lim_{n \to \infty} n s_n(A) = \frac{2}{\pi} h.$$

This can be established in the same way as the corresponding relation in Corollary 7.2.

6. Theorem 8.2, together with the formula (8.28), enables one to generalize result 1 of §7 to the case of a kernel $\mathscr{K}(t,s)$ which defines by the formula

$$(Hf)(t) = \int_a^b \mathscr{K}(t,s) f(s)\, ds$$

a Hermitian nuclear but not necessarily definite operator.

In this case, for a continuous kernel $\mathscr{K}(t,s)$ formula (7.16) can be replaced by the formula

$$\lim_{r\to\infty}\frac{n_\pm(r;G)}{r}=\frac{1}{\pi}\int_a^b \mathbf{sp}|\mathscr{K}(s,s)|\,ds,$$

where by $\mathbf{sp}|\mathscr{K}(s,s)|$ is understood the sum of the absolute values of the eigenvalues of the matrix $\mathscr{K}(s,s)$, and for a discontinuous kernel, by the formula

$$\lim_{r\to\infty}\frac{n_\pm(r;G)}{r}=\frac{1}{\pi}\lim_{\delta\to 0}\int_a^b \mathbf{sp}|\mathscr{K}_\delta(s,s)|\,ds,$$

where

$$\mathscr{K}_\delta(t,s)=\frac{1}{4\delta^2}\int_{t-\delta}^{t+\delta}\int_{s-\delta}^{s+\delta}\mathscr{K}(u,v)\,du\,dv.$$

We leave it to the reader to comment upon other relations from this section for the integral operators H and G being considered.

§9. An asymptotic property of the spectrum of an operator with nuclear imaginary component

1. We have already once used (see the proof of Theorem 8.1) the following result.[12)]

Let $u(\lambda)\,(\operatorname{Im}\lambda>0)$ be a positive harmonic function in the upper half-plane. Then

$$u(\lambda)=\left(h_u+\int_{-\infty}^{\infty}\frac{d\omega_u(t)}{|t-\lambda|^2}\right)\operatorname{Im}\lambda \qquad (\operatorname{Im}\lambda>0), \tag{9.1}$$

where

$$h_u=\inf_{\operatorname{Im}\lambda>0}\frac{u(\lambda)}{\operatorname{Im}\lambda}, \tag{9.2}$$

and $\omega_u(t)=\frac{1}{2}(\omega_u(t+0)+\omega_u(t-0))$ is a nondecreasing function, defined by the Stieltjes inversion formula:

$$\omega_u(t)=\lim_{h\to 0}\frac{1}{\pi}\int_0^t u(s+ih)\,ds.$$

12) This result is a simple corollary of a well-known theorem of Riesz-Herglotz (see I. I. Privalov [1], Chapter I, §2) on the integral representation of a function which is positive and harmonic in the unit disc.

Strictly speaking, the relation (9.2) is a consequence of the representation (9.1) itself; moreover, from this representation it follows that for any ϑ $(0 < \vartheta < \pi/2)$ the relation

$$\lim_{\rho\to\infty} \frac{u(\rho e^{i\theta})}{\rho} = h_u \sin\theta$$

holds uniformly in the sector

$$|\pi/2 - \theta| < \vartheta.$$

W. K. Hayman [1] has established an important theorem which, in particular, contains the following result.

1. *Let* $u(\lambda)$ *be a nonnegative superharmonic function for* $\operatorname{Im}\lambda > 0$, *and let*

$$h_u = \inf_{\operatorname{Im}\lambda > 0} \frac{u(\lambda)}{\operatorname{Im}\lambda}.$$

There always exists a set $\Delta \subset (1, \infty)$ *of finite logarithmic length such that the limit relation*

$$\lim_{\rho\to\infty} \frac{u(\rho e^{i\theta})}{\rho} = h_u \sin\theta$$

holds uniformly for all θ $(0 < \theta < \pi)$ *as* $\rho \to \infty$, *avoiding* Δ.

Let us clarify that the logarithmic length of a measurable set $\Delta \subset (1, \infty)$ is defined as the integral

$$\int_\Delta \frac{d\rho}{\rho}.$$

The result 1, in particular, is applicable to any nonnegative harmonic function $u(\lambda)$ $(\operatorname{Im}\lambda > 0)$. On the basis of it, we can assert for any product

$$B(\lambda) = \prod_j \frac{1 - \lambda/\mu_j}{1 - \lambda/\bar{\mu}_j} \quad \left(\sum_j \left|\operatorname{Im}\frac{1}{\mu_j}\right| < \infty, \ \mu_j \to \infty\right)$$

that

$$\lim \left[\frac{1}{\rho} \ln|B(\rho e^{i\theta})|\right] = 0, \tag{9.3}$$

uniformly in θ $(0 \leqq \theta \leqq \pi)$ as $\rho \to \infty$, avoiding an appropriate set $\Delta \subset (1, \infty)$ of finite logarithmic length.

In fact, putting

$$B_1(\lambda) = \prod_{\operatorname{Im}\mu_j<0} \frac{1-\lambda/\mu_j}{1-\lambda/\bar{\mu}_j}, \quad B_2(\lambda) = \prod_{\operatorname{Im}\mu_j>0} \frac{1-\lambda/\bar{\mu}_j}{1-\lambda/\mu_j},$$

we will have

$$\ln|B(\lambda)| = \ln|B_1(\lambda)| - \ln|B_2(\lambda)| = u_1(\lambda) - u_2(\lambda),$$

where $u_1(\lambda)$ and $u_2(\lambda)$ are superharmonic functions for $\operatorname{Im}\lambda > 0$. For these functions the constants h_{u_1} and h_{u_2} equal 0, since by the well-known properties of a Blaschke product (see M. G. Kreĭn [3]) there exists a sequence of numbers $\rho_n \to \infty$ such that the relations

$$\lim_{n\to\infty}\left[\frac{1}{\rho_n}\ln|B_j(\rho_n e^{i\theta})|\right] = \lim_{n\to\infty}\frac{u_j(\rho_n e^{i\theta})}{\rho_n} = 0 \qquad (j = 1,2)$$

will hold uniformly in θ $(0 < \theta \leqq 2\pi)$, and so, in particular,

$$h_{u_j} \leqq \inf_n \frac{u_j(i\rho_n)}{\rho_n} = 0 \qquad (j = 1,2).$$

2. With every completely continuous operator A we will associate the function

$$N(r;A) = \int_0^r \frac{n(\rho;A)}{\rho}\,d\rho \qquad (0 < r < \infty), \tag{9.4}$$

where $n(r;A)$ denotes the number of characteristic numbers of the operator A lying in the disc $|\lambda| \leqq r$.

Considerations similar to those by which result 1 of §14, Chapter III was established show that if for some $l > 0$

$$n(r;A) = r^l L(r) + o(r^l L(r)) \qquad (r\to\infty), \tag{9.5}$$

where $L(r)$ is a slowly varying function, then

$$N(r;A) = (1/l)r^l L(r) + o(r^l L(r)) \qquad (r\to\infty) \tag{9.6}$$

(see in this connection B. Ja. Levin [1], p. 50 (transl., p. 34)).

THEOREM 9.1. *Let $A = G + iH$ be a completely continuous operator with $A_{\mathscr{I}} = H \in \mathfrak{S}_1$, and let $\{\mu_j\}$ be its complete system of characteristic numbers. Then there exists a set $\Delta \subset (1,\infty)$ of finite logarithmic length such that for $r\to\infty$, avoiding Δ, one has*

$$N(r;G) - N(r;A) = \kappa r + o(r), \tag{9.7}$$

where

(9.8)
$$\kappa \geqq \frac{2}{\pi} \left| \operatorname{sp} H - \sum_j \operatorname{Im}\left(\frac{1}{\mu_j}\right) \right| .$$

If the operator A is dissipative, then

(9.9)
$$\kappa = \frac{2}{\pi} \left[\operatorname{sp} H - \sum_j \operatorname{Im}\left(\frac{1}{\mu_j}\right) \right] .$$

PROOF. Just as in the proof of Lemma 8.1, we shall use the identity

$$D_{G/A}(\lambda) = D_{G/A_1}(\lambda)\, D_{A_1/A}(\lambda) = \frac{D_{G/A_1}(\lambda)}{D_{A/A_1}(\lambda)},$$

where $A_1 = G + iH_1$ $(H_1 = H_+ + H_-)$. Let us recall that

$$|D_{G/A_1}(\lambda)| \leqq 1, \quad |D_{A/A_1}(\lambda)| \leqq 1 \qquad \text{for} \quad \operatorname{Im}\lambda \geqq 0,$$

and according to Lemma 4.2

(9.10)
$$\lim_{r\to\infty} \left[\frac{1}{r} \ln |D_{A_1/G}(ir)| \right] = 0.$$

Thus

$$\ln |D_{G/A}(\lambda)| = u_2(\lambda) - u_1(\lambda) \qquad (\operatorname{Im}\lambda > 0),$$

where

$$u_1(\lambda) = \ln |D_{A_1/G}(\lambda)| > 0 \qquad (\operatorname{Im}\lambda > 0),$$

and

$$u_2(\lambda) = \ln |D_{A_1/A}(\lambda)| = -\ln |D_{A/A_1}(\lambda)| > 0 \qquad (\operatorname{Im}\lambda > 0)$$

are superharmonic functions.

Let us denote by h_1^+ and h_2^+ the nonnegative numbers, and by Δ_1 and Δ_2 the sets of finite logarithmic length, which correspond to the functions $u_1(\lambda)$ and $u_2(\lambda)$ by Hayman's theorem. By virtue of (9.10) we will have $h_1^+ = 0$. Therefore, if we put $\Delta_+ = \Delta_1 \cup \Delta_2$, it follows on the basis of Hayman's theorem and the equality (9.10) that the limit relation

(9.11)
$$\lim \left[\frac{1}{r} \ln |D_{G/A}(re^{i\theta})| \right] = h_2^+ \sin\theta$$

holds uniformly with respect to θ $(0 \leqq \theta \leqq \pi)$ as $r \to \infty$, avoiding Δ_+.

Similarly, one can prove the existence of a constant $h_2 \geq 0$ and a set Δ of finite logarithmic length such that the limit relation

$$\lim\left[\frac{1}{r}\ln|D_{G/A}(re^{i\theta})|\right] = h_2^- \,|\sin\theta| \tag{9.12}$$

holds uniformly with respect to θ $(-\pi < \theta < 0)$ as $r \to \infty$, avoiding Δ_-. We shall show that

$$\frac{1}{2}\,(h_2^- - h_2^+) = \operatorname{sp} H - \sum_j \operatorname{Im}\left(\frac{1}{\mu_j}\right). \tag{9.13}$$

In fact, according to (9.11) and (9.12)

$$\lim\ln\frac{|D_{G/A}(ir)|}{|D_{G/A}(-ir)|} = h_2^+ - h_2^-$$

as $r \to \infty$, avoiding the set

$$\Delta = \Delta_+ \cup \Delta_-.$$

On the other hand, we have

$$\overline{D_{G/A}(-ir)} = D_{G/A^*}(ir), \qquad \frac{D_{G/A}(ir)}{D_{G/A^*}(ir)} = D_{A^*/A}(ir),$$

and according to Theorem 8.1 and the relation (9.3),

$$-\frac{1}{2}\lim\left[\frac{1}{r}\ln|D_{A^*/A}(ir)|\right] = \operatorname{sp} H - \sum_j \operatorname{Im}\left(\frac{1}{\mu_j}\right)$$

as $r \to \infty$, avoiding some set of finite logarithmic length. From this follows (9.13).

We now apply Hayman's formula to the meromorphic function $D_{G/A}(\lambda)$. Since the zeros and poles of this function coincide respectively with the characteristic numbers of the operators G and A, on the basis of this formula we will have

$$N(r;G) - N(r;A) = \frac{1}{2\pi}\int_0^{2\pi} \ln|D_{G/A}(re^{i\theta})|\,d\theta. \tag{9.14}$$

If $r \to \infty$, avoiding Δ, the asymptotic value of the integral in the right side of (9.14) can be calculated from the limit relations (9.11) and (9.12), which yields

$$N(r;G) - N(r;A) = \frac{1}{\pi}\,(h_2^+ + h_2^-)\,r + o(r) \tag{9.15}$$

as $r \to \infty$, avoiding Δ. Obviously, the first assertion of the theorem follows from (9.13) and (9.15), where $\kappa = (h_2^+ + h_2^-)/\pi$.

If the operator A is dissipative, then $A_1 = A$, $D_{A_1/A} = 1$, $h_2^+ = 0$, and according to (9.13)

$$h_2^- = 2\left[\operatorname{sp} H - \sum_j \operatorname{Im}\left(\frac{1}{\mu_j}\right)\right].$$

Thus, in this case $\kappa = (h_2^+ + h_2^-)/\pi$ will have the value (9.9). The theorem is proved.

Theorem 9.1 was established in a paper by M. G. Kreĭn [**9**]. Previous to this, B. Ja. Levin [2], the first to use Hayman's result, showed that in the case of a completely continuous *dissipative* operator $A = G + iH$ ($\operatorname{sp} H < \infty$) one has the asymptotic inequality

$$N(r; G) - N(r; A) \leqq r \operatorname{sp} H + o(r) \qquad (r \to \infty).$$

Little has to be added to the arguments of this paper in order to obtain, for the specified case of a dissipative operator, the asymptotic inequality $N(r; G) - N(r; A) \leqq \kappa r + o(r)$, where κ has the value indicated in (9.9).

§10. A theorem on Volterra operators with a finite-dimensional imaginary component

1. To establish this theorem we have to complete our arsenal of auxiliary function-theoretic results. In particular, we need the following generalization of Levinson's theorem (cf. §8, result B)), which was proved by M. Cartwright [1] (see B. Ja. Levin [1], Chapter V, §4, Theorems 7 and 11).

C) *Suppose that the entire function $f(\lambda)$, of not greater than exponential type, satisfies the condition*

$$\int_{-\infty}^{\infty} \frac{\ln^+ |f(\lambda)|\, d\lambda}{1+\lambda^2} < \infty.$$

Then for any ϑ $(0 < \vartheta < \pi)$ one has the limit relations

$$\lim_{r\to\infty} \frac{n_+(r;\vartheta)}{r} = \lim_{r\to\infty} \frac{n_-(r;\vartheta)}{r} \tag{10.1}$$
$$= \frac{1}{2\pi}\left\{ \overline{\lim_{r\to\infty}} \frac{\ln|f(ir)|}{r} + \overline{\lim_{r\to\infty}} \frac{\ln|f(-ir)|}{r} \right\},$$

where $n_+(r;\vartheta)$ and $n_-(r;\vartheta)$ denote, respectively, the number of zeros of $f(\lambda)$ in the sectors

$$|\lambda| \leqq r, \ |\arg \lambda| < \vartheta \ \textit{and} \ |\lambda| \leqq r, \ |\arg(-\lambda)| < \vartheta.$$

Moreover we need the following results.

D) *Every function* $F(\lambda)$ *of the form*

$$F(\lambda) = \sum_j \frac{m_j}{\alpha_j - \lambda} \tag{10.2}$$

belongs to the classes $(\mathfrak{A}_+)$ *and* $(\mathfrak{A}_-)$ [13] *in the respective halfplanes, where* $\{\alpha_j\}$ *is some sequence of real numbers and* $\{m_j\}$ *is a sequence of complex numbers such that*

$$\sum_j \frac{|m_j|}{1 + |\alpha_j|} < \infty. \tag{10.3}$$

This result is a corollary of a more general and precise result of V. I. Smirnov (see I. I. Privalov [1], Chapter II, §4). It admits a simple and direct proof. By virtue of the linearity of the classes $(\mathfrak{A}_\pm)$ it is sufficient to prove the result D) for the case in which $m_j > 0$ for all j $(j = 1, 2, \cdots)$. In this case we will have, for the function $\Phi(\lambda) = i + F(\lambda)$,

$$\operatorname{Im} \Phi(\lambda) = 1 + \sum_j \frac{m_j \operatorname{Im} \lambda}{|\alpha_j - \lambda|^2} > 1 \quad \text{for} \quad \operatorname{Im} \lambda > 0.$$

Therefore in the upper halfplane the modulus of the function $\Psi(\lambda) = 1/\Phi(\lambda)$ is bounded by unity, and consequently $\Phi(\lambda) = 1/\Psi(\lambda) \in (\mathfrak{A}_+)$ and $F(\lambda) = \Phi(\lambda) - i \in (\mathfrak{A}_+)$. Similarly, one proves that in the lower halfplane $F(\lambda) \in (\mathfrak{A}_-)$.

E) *Let* $F(\lambda)$ *be a meromorphic function of the form* (10.2), *where the additional condition*

$$\sum_{\alpha_j < 0} \frac{1}{\sqrt{|\alpha_j|}} < \infty \qquad (\sqrt{|\alpha_j|} > 0, j = 1, 2, \cdots) \tag{10.4}$$

is fulfilled. Let us put

$$B(\lambda) = \prod_{\alpha_j < 0} \frac{i\sqrt{|\alpha_j|} - \lambda}{i\sqrt{|\alpha_j|} + \lambda}. \tag{10.5}$$

Then in the upper halfplane $B(\lambda) F(\lambda^2) \in (\mathfrak{A}_+)$, *and in the lower halfplane* $B^{-1}(\lambda) F(\lambda^2) \in (\mathfrak{A}_-)$.

In fact, the function $F(\lambda^2)$ admits the representation [14]

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[13] For the definition of the classes $(\mathfrak{A}_\pm)$ see §8.1.

[14] Since $1/\lambda^2 \in (\mathfrak{A}_\pm)$, we may suppose without loss of generality that $\alpha_j \neq 0$, $j = 1, 2, \cdots$.

$$F(\lambda^2) = \sum_{\alpha_j > 0} \frac{m_j}{2\sqrt{\alpha_j}} \left(\frac{1}{\sqrt{\alpha_j} - \lambda} + \frac{1}{\sqrt{\alpha_j} + \lambda} \right)$$

$$+ \sum_{\alpha_j < 0} \frac{m_j}{2i\sqrt{|\alpha_j|}\,(i\sqrt{|\alpha_j|} + \lambda)} + \sum_{\alpha_j < 0} \frac{m_j}{2i\sqrt{|\alpha_j|}\,(i\sqrt{|\alpha_j|} - \lambda}$$

$$= \Phi_1(\lambda) + \Phi_2(\lambda) + \Phi_3(\lambda).$$

According to result D), $\Phi_1(\lambda) \in (\mathfrak{A}_+)$. Since for $\operatorname{Im}\lambda > 0$

$$\left| \frac{1}{i\sqrt{|\alpha_j|} + \lambda} \right| < \frac{1}{\sqrt{|\alpha_j|}}, \tag{10.6}$$

the function $\Phi_2(\lambda)$ is bounded for $\operatorname{Im}\lambda > 0$:

$$|\Phi_2(\lambda)| \leqq \frac{1}{2} \sum_{\alpha_j < 0} \frac{|m_j|}{|\alpha_j|} < \infty,$$

and consequently also belongs to the class $(\mathfrak{A}_+)$.

In the upper halfplane, the function $B(\lambda)$ is holomorphic and its modulus is bounded by unity, so that $B(\lambda) \in (\mathfrak{A}_+)$, and hence $B(\lambda)\,\Phi_p(\lambda) \in (\mathfrak{A}_+)$ $(p = 1, 2)$.

To prove that $B(\lambda)\,F(\lambda^2) \in (\mathfrak{A}_+)$, it remains to show that $B(\lambda)\,\Phi_3(\lambda) \in (\mathfrak{A}_+)$. To do this, we represent this function in the form

$$B(\lambda)\,\Phi_3(\lambda) = \frac{1}{2i} \sum_{\alpha_k < 0} \frac{m_k}{\sqrt{|\alpha_k|}} \frac{B(\lambda)}{i\sqrt{|\alpha_k|} - \lambda}$$

$$= \frac{1}{2i} \sum_{\alpha_k < 0} \frac{m_k}{\sqrt{|\alpha_k|}} \frac{B_k(\lambda)}{i\sqrt{|\alpha_k|} + \lambda}.$$

Since for $\operatorname{Im}\lambda > 0$

$$|B_k(\lambda)| = \left| \prod_{\alpha_j < 0; j \neq k} \frac{i\sqrt{|\alpha_j|} - \lambda}{i\sqrt{|\alpha_j|} + \lambda} \right| < 1,$$

it follows from (10.6) and (10.3) that

$$|B(\lambda)\,\Phi_3(\lambda)| \leqq \frac{1}{2} \sum_{\alpha_k < 0} \frac{|m_k|}{|\alpha_k|} < \infty \qquad (\operatorname{Im}\lambda > 0).$$

Thus $B(\lambda)\Phi_3(\lambda) \in (\mathfrak{A}_+)$, and so $B(\lambda)F(\lambda^2) \in (\mathfrak{A}_+)$. Similarly one can show that $B^{-1}(\lambda)\,F(\lambda^2) \in (\mathfrak{A}_-)$. Result E) is proved.

2. After all that we have set forth, the proof of the basic theorem is not difficult.

THEOREM 10.1. *Let $A = G + iH$ be a Volterra operator with a finite-dimensional imaginary component H. If the negative characteristic numbers α_j of the operator G satisfy the condition*

$$\sum_{\alpha_j<0} \frac{1}{\sqrt{|\alpha_j|}} < \infty, \tag{10.7}$$

then the finite limit

$$\lim_{r\to\infty} \frac{n_+(r;G)}{\sqrt{r}}$$

exists for its positive characteristic numbers.

We number the characteristic numbers so that $\alpha_j < 0$ for $j \leqq 0$ and $\alpha_j > 0$ for $j > 0$. The theorem asserts, in particular, that if G has an infinite number of positive characteristic numbers, then if we arrange them all in a nondecreasing sequence $\alpha_1 \leqq \alpha_2 \leqq \cdots$, the limit

$$\lim_{n\to\infty} \frac{\alpha_n}{n^2} \tag{10.8}$$

will exist.

PROOF. By hypothesis the component H admits the representation

$$H = \sum_{q=1}^{n} \epsilon_q(\cdot, \psi_q)\, \psi_q,$$

where $\psi_1, \cdots, \psi_n$ is some orthonormal system of vectors. Then

$$D_{A/G}(\lambda) = \det(I - i\lambda H(I-\lambda G)^{-1}) = \det\| \delta_{pq} - i\lambda\epsilon_q a_{pq}(\lambda) \|_1^n,$$

where

$$a_{pq}(\lambda) = ((I-\lambda G)^{-1}\psi_p, \psi_q) \qquad (q, p = 1, 2, \cdots, n).$$

Since A is a Volterra operator, the function $D_{G/A}(\lambda) = D_{A/G}^{-1}(\lambda)$ is entire with real zeros α_j $(j = 0, \pm 1, \pm 2, \cdots)$.

If we show that the function $f(\lambda) = D_{G/A}(\lambda^2)$ belongs to the classes $(\mathfrak{A}_\pm)$ in the respective halfplanes, then by result A) of §8, result C) will be applicable to the function $f(\lambda)$; since, on the other hand, for the function $f(\lambda)$ for $0 < \vartheta < \pi/2$, obviously, $n_+(\sqrt{r}; \vartheta) = n_+(r; G)$, the theorem will be proved.

Bearing this in mind, we introduce an orthonormal system of eigenvectors ϕ_j $(j = 0, \pm 1, \pm 2, \cdots)$ of the operator G, corresponding to the system of characteristic numbers α_j $(j = 0, \pm 1, \pm 2, \cdots)$, so that

$$\alpha_j G\phi_j = \phi_j \qquad (j = 0, \pm 1, \pm 2, \cdots).$$

Then

$$(I - \lambda G)^{-1} = I - \lambda \sum_j \frac{(\cdot, \phi_j)\, \phi_j}{\alpha_j - \lambda},$$

and so

$$a_{qp} = \delta_{qp} - \lambda \sum_j \frac{c_{qj}\bar{c}_{pj}}{\alpha_j - \lambda} \qquad (p, q = 1, 2, \cdots, n),$$

where $c_{qj} = (\phi_j, \psi_q)$.

Defining $B(\lambda)$ by (10.5), we will have, according to result E),

$$B(\lambda) \sum_j \frac{c_{qj}\bar{c}_{pj}}{\alpha_j - \lambda^2} \in (\mathfrak{A}_+) \qquad (p, q = 1, 2, \cdots, n),$$

and hence

$$B(\lambda)\, a_{qp}(\lambda^2) \in (\mathfrak{A}_+) \qquad (p, q = 1, 2, \cdots, n).$$

Consequently, if we multiply every row of the determinant

$$\det \| \delta_{pq} - i\lambda^2 \epsilon_q a_{pq}(\lambda^2) \|_1^n \quad (= D_{A/G}(\lambda^2))$$

by $B(\lambda)$, then all of its elements will belong to the class $(\mathfrak{A}_+)$. Since this class is an algebra of functions, it follows that

$$B^n(\lambda)\, D_{A/G}(\lambda^2) \in (\mathfrak{A}_+).$$

Taking into account that $B^{n-1}(\lambda)$, together with $B(\lambda)$, belongs to the class $(\mathfrak{A}_+)$, we conclude on the basis of property 2 (§8.1) of this class that the function

$$F(\lambda) = B(\lambda) D_{A/G}(\lambda^2) = \frac{B^n(\lambda)\, D_{A/G}(\lambda^2)}{B^{n-1}(\lambda)},$$

which is holomorphic in the upper halfplane, also belongs to this class. Since the function $F(\lambda)$ does not have zeros in the upper halfplane, the function $f(\lambda) = F^{-1}(\lambda)$, together with $F(\lambda)$, belongs to $(\mathfrak{A}_+)$, and consequently

$$D_{G/A}(\lambda^2) = B(\lambda)\, f(\lambda) \in (\mathfrak{A}_+).$$

One can similarly show that $D_{G/A}(\lambda^2) \in (\mathfrak{A}_-)$. Theorem 10.1 is proved.

Of course, one can interchange the roles of the positive and negative spectra of the characteristic numbers of the operator G in the statement of the theorem.

3. It is probable that the condition of finite-dimensionality of the component $H = A_{\mathscr{I}}$ in Theorem 10.1 can be weakened (cf. p. 221). However, even for a finite-dimensional component H Theorem 10.1 finds essential applications.[15]

It should not be thought that Theorem 10.1 can be made more precise by for example imposing more restrictive requirements concerning the *sparseness* of the negative spectrum of the operator $G = A_{\mathscr{R}}$.

Since, under the conditions of the theorem, the Volterra operator A turns out to belong to the class $\mathfrak{S}_1$ (moreover, to any class $\mathfrak{S}_p$ with $p > 1/2$), by Theorem III.8.4 one has $\operatorname{sp} H = 0$ and consequently $\operatorname{sp} G = 0$, so that G will always have eigenvalues of both signs.

We shall present below an example of a Volterra operator A with a *two-dimensional*[16] imaginary component $H = A_{\mathscr{I}}$ and a real component $G = A_{\mathscr{R}}$ which has exactly *one* negative eigenvalue and for which the limit (10.8) is *positive*.

We realize $\mathfrak{H}$ in the form $L_2(0,1)$ and consider the Volterra operator V in $L_2(0,1)$, defined by[17]

$$(Vf)(t) = \int_t^1 (t-s)\, f(s)\, ds \qquad (f \in L_2). \tag{10.9}$$

For this operator, obviously,

$$(V_{\mathscr{R}} f)(t) = -\frac{1}{2}\int_0^1 |t-s|\, f(s)\, ds, \tag{10.10}$$

$$(V_{\mathscr{I}} f)(t) = \frac{1}{2i}\int_0^1 (t-s)\, f(s)\, ds. \tag{10.11}$$

From (10.11) it is clear that the component $V_{\mathscr{I}}$ is two-dimensional.

To establish those properties of the spectrum of the operator $V_{\mathscr{R}}$ which are of interest to us, we shall use the identity

$$-\frac{1}{2}|t-s| = G(t,s) + \left(t-\frac{1}{2}\right)\left(s-\frac{1}{2}\right) - \frac{1}{4}, \tag{10.12}$$

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[15] See the authors' book [7].

[16] Since an operator with a one-dimensional imaginary component becomes dissipative when multiplied by ± 1, it follows from the theorem of M. S. Livšic, cited on p. 187, that the condition (10.7) can never be fulfilled for such an operator (cf. also, in this connection, the general Theorem 7.2).

[17] This example was also considered by V. B. Lidskiĭ [5].

where

$$G(t,s) = \begin{cases} t(1-s) & (t \leqq s), \\ s(1-t) & (s \leqq t). \end{cases}$$

The integral equation

$$\phi = \mu\Gamma\phi \qquad \left((\Gamma\phi)(t) = \int_0^1 G(t,s)\,\phi(s)\,ds\right)$$

is equivalent to the boundary value problem

$$\phi'' + \mu\phi = 0, \qquad \phi(0) = \phi(1) = 0,$$

whose spectrum consists of the numbers

$$\mu_n = \pi^2 n^2 \qquad (n = 1, 2, \cdots). \tag{10.13}$$

On the other hand, it follows from (10.12) that

$$V_{\mathscr{R}} = \Gamma + (\cdot, e_1)\,e_1 - (\cdot, e_0)\,e_0,$$

where $e_1(t) = t - 1/2$ and $e_0(t) = 1/2$. Thus the operator $V_{\mathscr{R}}$ is obtained from the positive operator $\Gamma + (\cdot, e_1)\,e_1$ by the subtraction of a one-dimensional operator, and consequently it has at most one negative eigenvalue (this easily follows from Lemma II.1.2). Since sp $V_{\mathscr{R}} = 0$, $V_{\mathscr{R}}$ has exactly one negative eigenvalue. Taking (10.13) into account and recalling Corollary II.2.1, we conclude that for the sequence $\alpha_1 \leqq \alpha_2 \leqq \cdots$ of positive characteristic numbers of the operator $V_{\mathscr{R}}$ we have the asymptotic formula [18)]

$$\lim_{n\to\infty} \frac{\alpha_n}{n^2} = \pi^2.$$

§11. Further theorems on relations between the Hermitian components of Volterra operators

1. The question, to what extent do various properties of one of the components $A_{\mathscr{R}}$, $A_{\mathscr{J}}$ of a Volterra operator A determine properties of the other component, plays a fundamental role.

In the previous sections this question was studied for the case in which the imaginary component of the Volterra operator was nuclear

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[18)] It is not difficult to show that the spectrum of characteristic numbers of the operator $V_{\mathscr{R}}$ coincides with the spectrum of the boundary value problem

$$\phi'' + \mu\phi = 0,\ \phi'(0) + \phi'(1) = \phi(0) + \phi(1) + \phi'(0) = 0.$$

(§§6—9) or finite-dimensional (§10). In this section we shall present results concerning other important cases. All of these were obtained by V. I. Macaev [3] by using new theorems of the theory of entire functions, established by him, in the theory of perturbation determinants.

In particular we shall use below the following theorems on entire functions, which were announced by V. I. Macaev, among other theorems, in his communications [1-3].

I) *Let* $F(r)$ $(0 \leqq r < \infty)$ *be a nondecreasing continuously differentiable function which satisfies the conditions* $F(0) = 1$, $F'(0) = 0$, *and also, for some* p $(1 < p < 2)$, *the condition*

$$\int_0^\infty \frac{\ln F(r)}{r^{1+p}}\, dr < \infty .$$

Then if an entire function $f(z)$ *admits the lower bound*

$$|f(z)| \geqq 1/F\left(\frac{r}{|\sin\theta|}\right) \qquad (z = re^{i\theta}),$$

the inequality

$$\sum_j \frac{1}{|a_j|^p} \leqq C_p \int_0^\infty \frac{\ln F(r)}{r^{1+p}}\, dr$$

will be fulfilled for the sequence $\{a_j\}$ *of all the zeros of* $f(z)$, *where* C_p *is a constant depending only upon* p.

II) *If the entire function* $f(z)$ *admits, for some* $\rho > 1$ *and* $C > 0$, *the lower bound*

$$|f(z)| \geqq C \exp\left\{ - \left|\frac{r}{\sin\theta}\right|^\rho L_1\left(\frac{r}{|\sin\theta|}\right)\right\},$$

where $L_1(r)$ *is a slowly varying function, then*

$$\ln^+|f(z)| = O(r^\rho L_1(r)) \qquad (r \to \infty). \tag{11.1}$$

We remark that on the basis of a well-known theorem of Lindelöf-Valiron (see B. Ja. Levin [1], Chapter I, Theorem 17), for noninteger ρ the relation (11.1) is equivalent to

$$n(r;f) = O(r^\rho L_1(r)) \qquad (r \to \infty), \tag{11.2}$$

where $n(r;f)$ $(0 < r < \infty)$ is the number of zeros of the function $f(z)$ lying in the disc $|z| \leqq r$.

Assuming that the zeros $\{a_j\}$ of the entire function $f(z)$ are numbered in order of nondecreasing modulus $|a_1| \leqq |a_2| \leqq \cdots$, we can assert that the last relation, in turn, is equivalent to

$$\frac{1}{|a_n|} = O(n^{-1/\rho} L(n)) \qquad (n \to \infty), \tag{11.2'}$$

where $L(r)$ is a slowly varying function, determined from the condition that the functions $\phi_1(r) = r^\rho L_1(r)$ and $\phi_2(r) = r^{1/\rho} L(r)$ are, for sufficiently large r, inverses of each other, i.e. $\phi_1(\phi_2(r)) = \phi_2(\phi_1(r)) = r$ for $r > R$.[19] The equivalence of the relations (11.2) and (11.2') is easily verified, if one notes that a positive slowly varying function $L(r)$ satisfies, for any $a > 0$, the condition $L(ar)/L(r) \to 1$ as $r \to \infty$.

III) *If the entire function $f(z)$ admits, for some positive $\rho < 1$ (and $C > 0$), the lower bound*

$$|f(z)| \geqq C \exp\left(-\left|\frac{r}{\sin\theta}\right|^\rho\right),$$

then

$$\ln|f(z)| = O(|z|) \quad \textit{for } z \to \infty \textit{ and } \int_{-\infty}^{\infty} \frac{|\ln|f(t)||}{1+t^2}\, dt < \infty,$$

and consequently for the zeros of $f(z)$, arranged in a nondecreasing sequence $\{a_j\}_{-\infty}^{\infty}$, the finite limits

$$\lim_{n\to\infty} \frac{n}{a_n} \qquad \textit{and} \qquad \lim_{n\to-\infty} \frac{n}{a_n}$$

exist and are equal.

The last conclusion was made on the basis of a theorem of Cartwright-Levinson (cf. result B) of §8).

2. We also need the following elementary lemma:

Lemma 11.1. *Let $a_1 \geqq a_2 \geqq \cdots$ be a sequence of positive numbers such that $\sum_j a_j < \infty$. Put*

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[19] The existence, for sufficiently large r, of an inverse function $\phi_2(r)$ of the required form is seen from the following considerations. If we put $\xi = \ln r$, $\eta = \ln \phi_1(r)$, then the condition $rL_1'(r)/L_1(r) \to 0$ as $r \to \infty$ is equivalent to the condition $d\eta/d\xi \to \rho$ as $\xi \to \infty$. By virtue of the latter, $d\xi/d\eta \to 1/\rho$ as $\eta \to \infty$, from which follows the existence of an inverse function $\phi_2(r)$ of the form $r^{1/\rho} L(r)$.

$$\Pi(r) = \prod_{j=1}^{\infty} (1 + a_j r).$$

Then for any p $(0 < p < 1)$

$$\int_0^\infty \frac{\ln \Pi(r)}{r^{p+1}}\, dr = \beta_p \sum_{j=1}^{\infty} a_j^p \quad (\leqq \infty), \tag{11.3}$$

where

$$\beta_p = \frac{1}{p} \int_0^\infty \frac{dt}{t^p(1+t)} = \frac{\pi}{p \sin p\pi}.$$

If the series in (11.3) *has a finite value, then*

$$\ln \Pi(r) = o(r^p) \text{ for } r \to \infty.$$

PROOF. In fact,

$$\frac{\Pi'(r)}{\Pi(r)} = \sum_{j=1}^{\infty} \frac{a_j}{1 + a_j r},$$

and consequently for any p $(0 < p < 1)$

$$\int_0^\infty \frac{d \ln \Pi(r)}{r^p} = \int_0^\infty \frac{\Pi'(r)}{r^p \Pi(r)}\, dr = \sum_{j=1}^{\infty} \int_0^\infty \frac{a_j}{r^p(1 + a_j r)}\, dr = p\beta_p \sum_{j=1}^{\infty} a_j^p.$$

Since

$$\int_0^R \frac{d \ln \Pi(r)}{r^p} = \frac{\ln \Pi(R)}{R^p} + p \int_0^R \frac{\ln \Pi(r)}{r^{p+1}}\, dr,$$

it follows that if the left side tends to a finite limit as $R \to \infty$, then both terms on the right side will tend to a finite limit, and consequently the first of them will tend to zero.

Thus if the series in (11.3) has a finite value,

$$\int_0^\infty \frac{\ln \Pi(r)}{r^{p+1}} dr = p^{-1} \int_0^\infty \frac{d \ln \Pi(r)}{r^p} = \beta_p \sum_{j=1}^{\infty} a_j^p.$$

Since, by virtue of this equality, for any integer n

$$\sum_{j=1}^{n} a_j^p = \frac{1}{\beta_p} \int_0^\infty \frac{\ln \Pi_n(r)}{r^{p+1}}\, dr$$

$$\leqq \frac{1}{\beta_p} \int_0^\infty \frac{\ln \Pi(r)}{r^{p+1}}\, dr \qquad \left(\Pi_n(r) = \prod_1^n (1 + a_j r)\right),$$

the lemma is proved.

3. Now let $A = G + iH$ be a Volterra operator with imaginary component $H \in \mathfrak{S}_p$, where $1 < p < 2$. For any F-regular point μ of the operator G we introduce the operators

$$\Gamma(\mu) = (I - \mu G)^{-1}, \quad C(\mu) = -(\Gamma(\mu) H)^2.$$

Since

$$|C(\mu)|_1 \leqq |C(\mu)|_{p/2} \leqq |\Gamma(\mu) H|_p^2 \leqq |\Gamma(\mu)|^2 |H|_p^2 < \infty,$$

it follows that $C(\mu)$ is a holomorphic operator-function (with values in $\mathfrak{S}_1$) in the region $\mathfrak{G}$ of all F-regular points of the operator G. Consequently, the determinant

$$\Delta(\mu) = \det(I - \mu^2 C(\mu))$$

is a holomorphic function in the region $\mathfrak{G}$.

Since the operator $I - \mu^2 C(\mu)$ admits the decomposition

$$I - \mu^2 C(\mu) = \Gamma(\mu)(I - \mu A)\,\Gamma(\mu)(I - \mu A^*) \qquad (\mu \in \mathfrak{G}), \tag{11.4}$$

it has a bounded inverse for any $\mu \in \mathfrak{G}$:

$$\begin{aligned}(I - \mu^2 C(\mu))^{-1} & \\ &= (I - \mu A^*)^{-1}(I - \mu G)(I - \mu A)^{-1}(I - \mu G) \qquad (\mu \in \mathfrak{G}),\end{aligned}$$

and consequently

$$\Delta(\mu) \neq 0 \qquad (\mu \in \mathfrak{G}).$$

We will show that every point $\mu_j = \lambda_j^{-1}(G)$ is a pole of the determinant $\Delta(\mu)$ of multiplicity $2k$, equal to twice the multiplicity of $\lambda_j(G)$ as an eigenvalue of the operator G. This can be done starting from the same idea which lay at the basis of the proof of result 4 in §3.

Let us denote by P_j $(j = 1, 2, \cdots)$ the orthogonal projector on the eigenspace of the operator G corresponding to the eigenvalue $\lambda_j(G)$, and by G_j the operator $(I - P_j)\, G(I - P_j)$. As is known, $G = G_j + \mu_j^{-1} P_j$, and the point μ_j is an F-regular point of the operator G_j. It is easily seen that

$$\begin{aligned}\Gamma(\mu) &= (I - \mu G_j)^{-1}\left(I - \frac{\mu}{\mu_j} P_j\right)^{1} \\ &= \left(I - \frac{\mu}{\mu_j} P_j\right)^{-1} (I - \mu G_j)^{-1}.\end{aligned} \tag{11.5}$$

Since

$$\left(I - \frac{\mu}{\mu_j} P_j\right)^{-1} = I - \frac{\mu}{\mu - \mu_j} P_j \quad (= M_j),$$

we have

$$\det \left(I - \frac{\mu}{\mu_j} P_j\right)^{-1} = \det M_j = \frac{\mu_j^k}{(\mu_j - \mu)^k}.$$

It follows from (11.4) and (11.5) that in a sufficiently small neighborhood $\mathfrak{V}_j$ of the point μ_j $(\mu \neq \mu_j)$

$$\Delta(\mu) = \det(M_j S_j M_j R_j),$$

where

$$S_j = (I - \mu G_j)^{-1}(I - \mu A) \quad \text{and} \quad R_j = (I - \mu G_j)^{-1}(I - \mu A^*).$$

The operators $S_j M_j R_j - I$ and $M_j - I$ belong to $\mathfrak{S}_1$; consequently, according to result 7 of §1

$$\Delta(\mu) = \frac{\mu_j^k}{(\mu_j - \mu)^k} \det(S_j M_j R_j) \qquad (\mu \in \mathfrak{V}_j).$$

The operator S_j has a bounded inverse for all $\mu \in \mathfrak{V}_j$; consequently, according to result 6 of §1

$$\det(S_j M_j R_j) = \det(S_j^{-1} S_j M_j R_j S_j) = \frac{\mu_j^k}{(\mu_j - \mu)^k} \det(R_j S_j).$$

Thus

$$\Delta(\mu) = \frac{\mu_j^{2k}}{(\mu_j - \mu)^{2k}} \det(R_j S_j).$$

Since the operators R_j and S_j have bounded inverses for $\mu = \mu_j$ and $R_j S_j - I \in \mathfrak{S}_1$, it follows that

$$\det(R_j S_j)\,|_{\mu=\mu_j} \neq 0.$$

Consequently the determinant $\Delta(\mu)$ has a pole of multiplicity $2k$ at the point $\mu = \mu_j$. Thus we have shown that the function

(11.6) $$f(\mu) = \frac{1}{\Delta(\mu)}$$

is entire with zeros $\mu_j = \lambda_j^{-1}(G)$ $(j = 1, 2, \cdots)$.

By virtue of the bound (1.2) we have

(11.7) $$|\Delta(\mu)| \leqq \prod_j (1 + |\mu|^2 s_j(C(\mu))).$$

Taking into account that according to the bound (II.4.11)

$$\sum_{j=1}^k s_j(C(\mu)) = \sum_{j=1}^k s_j[(\Gamma(\mu) H)^2] \leqq \sum_{j=1}^k s_j^2(\Gamma(\mu) H) \qquad (k = 1, 2, \cdots),$$

we obtain, on the basis of Corollary II.3.1,

$$|\Delta(\mu)| \leqq \prod_{j=1}^\infty (1 + |\mu|^2 s_j^2(\Gamma(\mu) H)).$$

Since $\mu\Gamma(\mu) = (\lambda I - G)^{-1}$ $(\lambda = \mu^{-1})$, and for the selfadjoint operator G one has

$$|(\lambda I - G)^{-1}| \leqq \frac{1}{|\operatorname{Im} \lambda|} \qquad (\operatorname{Im} \lambda \neq 0),$$

it follows that

$$s_j(\Gamma(\mu) H) \leqq |\Gamma(\mu)| s_j(H) \leqq \frac{|\mu|}{|\operatorname{Im} \mu|} s_j(H) \qquad (j = 1, 2, \cdots).$$

Thus, by virtue of (11.7)

(11.8) $$|f(\mu)| \geqq 1\Big/F\left(\frac{r}{|\sin\theta|}\right) \qquad (\mu = re^{i\theta}),$$

where

$$F(r) = \prod_{j=1}^\infty (1 + s_j^2(H) r^2).$$

According to Lemma 11.1

$$I_p = \int_0^\infty \frac{\ln F(r)}{r^{p+1}} dr = \frac{1}{2}\int_0^\infty \frac{\ln F(\sqrt{r})}{r^{p/2+1}} dr = \frac{\beta_{p/2}}{2} \sum_{j=1}^\infty s_j^p(H) < \infty,$$

and therefore, according to the result I),

$$\sum_{j=1}^\infty \frac{1}{|\mu_j(G)|^p} = \sum_{j=1}^\infty |\lambda_j(G)|^p \leqq C_p \frac{\beta^{p/2}}{2} |H|_p^p.$$

Thus for $1 < p < 2$ we have obtained the following theorem.

THEOREM 11.1 (V. I. MACAEV [3]). *Let A be a Volterra operator. If its imaginary component $A_{\mathcal{J}}$ belongs to $\mathfrak{S}_p$ $(1 < p < \infty)$, then its real component $A_{\mathcal{R}}$ belongs to $\mathfrak{S}_p$, and moreover*

$$(11.9) \qquad |A_{\mathscr{R}}|_p \leqq \gamma_p |A_{\mathscr{I}}|_p,$$

where γ_p is a constant depending only upon p.

For $p = 2$ this theorem was first proved by L. A. Sahnovič [2] with the sharpened conclusion that $|A_{\mathscr{R}}|_2 = |A_{\mathscr{I}}|_2$. A simple proof of Sahnovič's theorem was given by the authors [**2**]. From the authors' results [**4**] it also followed that it suffices to prove the theorem for one of the two half-open intervals $(1, 2]$ and $[2, \infty)$, and that the exact constant γ_p in the bound (11.9), as a function of $1/p$, has the property of logarithmic convexity and the symmetry property $\gamma_p = \gamma_q$ $(p^{-1} + q^{-1} = 1)$.

In the authors' communication [4] one can find a proof of this theorem, for all $p > 1$, which does not require new or strong techniques of the theory of functions, and also a derivation of a simple bound for γ_p: $\gamma_p \leqq (e^{2/3} \ln 2)^{-1} p$ $(p \geqq 2)$.

A detailed presentation of these results will be given in the authors' book [7]. It will not be given here, since it is based on the theory of the abstract triangular integral.

4. We also have the following

THEOREM 11.2 (MACAEV [3], GOHBERG AND KREĬN [4]). *Let A be a Volterra operator. If*

$$(11.10) \qquad s_n(A_{\mathscr{I}}) = O(n^{-1/p} L(n)) \qquad (n \to \infty),$$

where $1 < p < \infty$ and $L(r)$ is a slowly varying function, then also

$$(11.11) \qquad s_n(A_{\mathscr{R}}) = O(n^{-1/p} L(n)) \qquad (n \to \infty).$$

The theorem remains valid if in (11.10) and (11.11) we replace O by o.

For values of p from the interval $1 < p < 2$ this theorem was proved by V. I. Macaev [3]. The authors' method, based on the theory of the abstract triangular integral, enables one to show that Theorem 11.2 is valid for all p $(1 < p < \infty)$, once the validity of the first assertion of the theorem has been proved for all values of p from the interval $1 < p < 2$.

We have to restrict ourselves to the discussion of the proof of the first assertion of the theorem for the case $1 < p < 2$.

PROOF. In fact, if the condition (11.10) is fulfilled for $H = A_{\mathscr{I}}$, then $H \in \mathfrak{S}_q$ for $q > p$. Since it is always possible to choose q such that $p < q < 2$, the entire functions $f(\mu)$ and $F(r)$ are defined and fulfill the relation (11.8). By the theorem of Lindelöf-Valiron (see §11.1) we will have, for the function $F(r)$,

$$\ln F(r) = O(r^p L_1(r)) \qquad (r \to \infty),$$

where $L_1(r)$ is some slowly varying function, whose relationship with the function $L(r)$ we already know.

Then by (11.8), for some $C > 0$,

$$\ln \frac{1}{|f(\mu)|} \leqq C \left(\frac{r}{|\sin\theta|} \right)^p L_1 \left(\frac{r}{|\sin\theta|} \right) \qquad (\mu = re^{i\theta}),$$

and, consequently, according to the result II), also

$$\ln|f(\mu)| = O(r^p L_1(r)) \qquad (r \to \infty).$$

Hence on the basis of the Lindelöf-Valiron theorem we obtain

$$|\lambda_n(G)| = O(n^{-1/p} L(n)) \qquad (n \to \infty; G = A_{\mathfrak{R}}).$$

The theorem is proved.

5. It is essential in Theorem 11.1 that $p > 1$. In fact, if the imaginary component $A_{\mathfrak{J}}$ of a dissipative Volterra operator A belongs to $\mathfrak{S}_p$ for some $p \leqq 1$ (or even if the component $A_{\mathfrak{J}}$ is finite-dimensional and consequently belongs to the class $\mathfrak{S}_p$ for arbitrarily small $p > 0$), by Theorem 7.2 the real component $A_{\mathfrak{R}}$ will not even belong to the class $\mathfrak{S}_1$. However, if to the condition $A_{\mathfrak{J}} \in \mathfrak{S}_p$ $(\frac{1}{2} < p < 1)$ we add the condition $A_{\mathfrak{R}}^+ \in \mathfrak{S}_p$ or $A_{\mathfrak{R}} \in \mathfrak{S}_p$ ($A_{\mathfrak{R}}^+$ and $A_{\mathfrak{R}}^-$ are the mutually orthogonal nonnegative operators from the decomposition $A_{\mathfrak{R}} = A_{\mathfrak{R}}^+ - A_{\mathfrak{R}}^-$), then it will follow that the Volterra operator A belongs to $\mathfrak{S}_p$. It will be convenient for us to formulate this result in the following form.

THEOREM 11.3 (MACAEV [3]). *If the Volterra operator A can be represented in the form*

$$A = C + T,$$

where $T \in \mathfrak{S}_p$ $(\frac{1}{2} < p < 1)$ and C is a nonnegative operator, then $A \in \mathfrak{S}_p$ and so $C \in \mathfrak{S}_p$, and moreover

$$|C|_p \leqq \kappa_p |T|_p, \tag{11.12}$$

where κ_p is a constant depending only upon p.

The theorem is also valid for $p = 1$ in the sharper form that $\kappa_p = 1$, and moreover $\operatorname{sp} C \leqq \operatorname{sp} T_{\mathfrak{R}}^-$. This result was noted in Remark IV.8.2.

PROOF. Let us consider the perturbation determinant $D_{A/C}(\mu)$. Replacing μ by μ^2, we obtain

$$D_{A/C}(\mu^2) = \det(I + N(\mu)),$$

where

$$N(\mu) = -\mu^2 T(I - \mu^2 C)^{-1} = -T\left(\frac{1}{\mu} I - C^{1/2}\right)^{-1}\left(\frac{1}{\mu} I + C^{1/2}\right)^{-1}. \tag{11.13}$$

We shall denote by $C^{1/2}$ the *nonnegative* operator whose square is the operator C. However, it is essential for us only that $C^{1/2}$ be a self-adjoint operator, in consequence of which

$$\left|\left(\frac{1}{\mu} I \pm C^{1/2}\right)^{-1}\right| \leqq \frac{1}{|\operatorname{Im}(1/\mu)|} = \frac{|\mu|^2}{|\operatorname{Im}\mu|} = \frac{r}{|\sin\theta|} \qquad (\mu = re^{i\theta}). \tag{11.14}$$

According to the bound (11.2),

$$|D_{A/C}(\mu^2)| \leqq \prod_{j=1}^{\infty}(1 + s_j(N(\mu))).$$

On the other hand, according to (11.13) and (11.14)

$$s_j(N(\mu)) \leqq \frac{r^2}{\sin^2\theta} s_j(T) \qquad (j = 1, 2, \cdots),$$

so that

$$\frac{1}{|D_{C/A}(\mu^2)|} = |D_{A/C}(\mu^2)| \leqq F\left(\frac{r}{|\sin\theta|}\right), \tag{11.15}$$

where

$$F(r) = \prod_{j=1}^{\infty}(1 + r^2 s_j(T)). \tag{11.16}$$

According to Lemma 11.1, we obtain

$$\int_0^\infty \frac{\ln F(r)}{r^{2p+1}}\,dr = \frac{1}{2}\int_0^\infty \frac{\ln F(\sqrt{r})}{r^{p+1}}\,dr = \frac{\beta_p}{2}\sum_{j=1}^{\infty} s_j^p(T) = \frac{\beta_p}{2}|T|_p^p < \infty.$$

Consequently Theorem I) (with the notational difference that p is replaced by $2p$) is applicable to the function $f(\mu) = D_{C/A}(\mu^2) = \det[(I - \mu^2 C)(I - \mu^2 A)^{-1}]$. Since the zeros of the function $f(\mu)$ are the numbers $\alpha_{\pm j} = \pm\lambda_j^{-1/2}(C)$ $(j = 1, 2, \cdots)$, by Theorem I) we obtain

$$|C|_p^p = \sum_{j=1}^{\infty} \lambda_j^p(C) \leq \frac{1}{4} C_p \beta_p |T|_p^p.$$

The theorem is proved.

COROLLARY 11.1. *Let $A = A_{\mathscr{R}} + iA_{\mathscr{I}}$ be a Volterra operator with a finite-dimensional imaginary component $A_{\mathscr{I}}$, and let $\{\lambda_j\}$ be the complete system of eigenvalues of its real component $A_{\mathscr{R}}$. Then for any p from the interval $\frac{1}{2} < p < 1$, the two series*

$$\sum_{\lambda_j < 0} |\lambda_j|^p, \qquad \sum_{\lambda_j > 0} \lambda_j^p$$

converge or diverge together.

To obtain this corollary, one uses Theorem 11.3 as applied to $C = A_{\mathscr{R}}^+$ and $T = -A_{\mathscr{R}}^- + iA_{\mathscr{I}}$ or to $C = A_{\mathscr{R}}^-$ and $T = -A_{\mathscr{R}}^+ - iA_{\mathscr{I}}$, where $A_{\mathscr{R}}^+$ and $A_{\mathscr{R}}^-$ are the orthogonal nonnegative operators from the decomposition $A_{\mathscr{R}} = A_{\mathscr{R}}^+ - A_{\mathscr{R}}^-$.

A comparison of Corollary 11.1 with Theorem 10.1 shows that this corollary, and so, more generally, Theorem 11.3, ceases to be true for $p \leq 1/2$.

6. Applying Theorem II) instead of Theorem I) to the function $f(\mu) = D_{C/A}(\mu^2)$ enables us to obtain the following result.

THEOREM 11.4 (MACAEV [3]). *If in the representation $A = C + T$ of the Volterra operator A the operator C is nonnegative, and the operator T $(\in \mathfrak{S}_\infty)$ is such that for some p $(\frac{1}{2} < p < 1)$ and some slowly varying function $L(r)$*

$$s_n(T) = O(n^{-1/p} L(n)) \qquad (n \to \infty), \tag{11.17}$$

then

$$\lambda_n(C) = O(n^{-1/p} L(n)) \qquad (n \to \infty). \tag{11.18}$$

PROOF. In fact, if the condition (11.17) is fulfilled, then $T \in \mathfrak{S}_1$ and consequently the entire function $f(\mu) = D_{C/A}(\mu^2)$ is meaningful, and one has the bound (11.15), where $F(r)$ is defined by (11.16). On the basis of (11.17), by the Lindelöf-Valiron theorem (cf. §11.1) we will have

$$\ln F(r) = O(r^{2p} L_1(r)),$$

where $L_1(r)$ is some positive slowly varying function. Therefore by virtue of (11.15)

$$\ln\frac{1}{|f(\mu)|} \leqq C\left(\frac{r}{|\sin\theta|}\right)^{2p} L_1\left(\frac{r}{|\sin\theta|}\right) \qquad (\mu = re^{i\theta}),$$

which by virtue of Theorem II) yields, in turn, the estimate

$$\ln|f(\mu)| = O(r^{2p}L_1(r)) \qquad (r = |\mu| \to \infty).$$

Recalling that the only zeros of the function $f(\mu)$ are the numbers $\pm \lambda_n^{1/2}(H)$, we conclude on the basis of the theorem of Lindelöf-Valiron that (11.18) holds.

COROLLARY 11.2. *Let $A \in \mathfrak{S}_\infty$ be a Volterra operator with a finite-dimensional imaginary component, and let $\lambda_1^+ \geqq \lambda_2^+ \geqq \cdots$ and $\lambda_1^- \leqq \lambda_2^- \leqq \cdots$ be respectively the complete systems of positive and negative eigenvalues of the real component $A_{\mathfrak{R}}$. Then for any p ($\frac{1}{2} < p < 1$) and any slowly varying function $L(r)$, the relation*

$$\lambda_n^- = O(n^{-1/p}L(n)) \quad (n \to \infty) \tag{11.19}$$

implies the relation

$$\lambda_n^+ = O(n^{-1/p}L(n)) \qquad (n \to \infty). \tag{11.20}$$

Indeed, if $A_{\mathfrak{J}}$ is an r-dimensional operator, then for the s-numbers of $A_{\mathfrak{R}}^-$ and $T = -A_{\mathfrak{R}}^- + iA_{\mathfrak{J}}$ we will have, according to Corollary II.2.1,

$$s_{n+r}(T) \leqq s_n(A_{\mathfrak{R}}^-) = |\lambda_n^-(A_{\mathfrak{R}}^-)| \leqq s_{n-r}(T) \qquad (n = r, r+1, \cdots),$$

in consequence of which (11.17) will follow from (11.19).[20]

Thus Theorem 11.4 is applicable to the operator $A = C + T$ ($C = A_{\mathfrak{R}}^+$), which yields (11.20).

THEOREM 11.5 (MACAEV [3]). *If in the representation $A = C + T$ of the Volterra operator A the operator C is nonnegative, and the operator T belongs to $\mathfrak{S}_p$ for some $p < 1/2$, then the limit $\lim_{n\to\infty} n^2\lambda_n(C)$ exists and is finite.*

PROOF. In fact, if $T \in \mathfrak{S}_p$ ($0 < p < 1/2$), then according to Lemma 11.1

$$\ln F(r) = \ln\prod_{j=1}^{\infty}(1 + r^2 s_j(T)) = o(r^{2p}) \qquad (r \to \infty).$$

[20] We have used here an easily proved property of slowly varying functions, namely that for any $l > 0$ one has $L(r+l)/L(r) \to 1$ as $r \to \infty$.

Consequently, according to the estimate (11.15) we will have, for the entire function $f(\mu) = D_{C/A}(\mu^2)$,

$$|f(\mu)| \geqq C \exp\left(-\left|\frac{r}{\sin\theta}\right|^{2p}\right) \qquad (\mu = re^{i\theta}).$$

Thus the result III) is applicable to the function $f(\mu)$, from which we obtain the conclusion of Theorem 11.5.

Combining Theorems 10.1 and 11.5 makes natural the following conjecture, which was stated by M. G. Kreĭn and V. I. Macaev.

If the Volterra operator A admits the representation $A = C + T$, where the operator C is nonnegative and $T \in \mathfrak{S}_{1/2}$, then the limit $\lim_{n\to\infty} n^2\lambda_n(C)$ exists and is finite.

CHAPTER V

THEOREMS ON THE COMPLETENESS OF THE SYSTEM OF ROOT VECTORS

In this chapter we discuss a number of tests for the completeness of the system of root vectors of a completely continuous nonselfadjoint operator. In our choice of material we have been guided by the desire, on the one hand, to collect rather original and strong tests and, on the other hand, to present as fully as possible the various methods for establishing such tests.

Roughly, these methods can be divided into three groups: 1) methods based on the study of the behavior and growth of the resolvent operator; 2) methods connected with the idea of the triangular representation of an operator, and 3) methods based on the analytic apparatus of the theory of perturbation determinants.

Particularly interesting tests are obtained when one brings together methods from the different groups.

In applications one encounters questions concerning the n-fold completeness of the system of eigenvectors and associated vectors of polynomial operator bundles. A special section is devoted to this question.

In §10 we indicate how tests for the completeness of the system of root vectors of an unbounded operator can be obtained from tests for completeness for completely continuous operators.

In the next to last section we discuss theorems on the asymptotic behavior of the eigenvalues of nonselfadjoint operators. The last section is devoted to investigations in the theory of selfadjoint quadratic bundles; this section uses the results of almost all of the preceding sections.

This chapter apparently contains the first adequately detailed treatment of the fundamental results of M. V. Keldyš [1] on the theory of nonselfadjoint operators.

§ 1. Lemmas on dissipative operators.

1. Let us recall once again that a bounded operator A is said to be *dissipative* if its imaginary component $A_{\mathcal{J}} = (A - A^*)/2i$ is a nonnegative operator.

Generalizing the definition of a simple Volterra operator, given in §7.4, Chapter IV, let us agree to call a bounded operator $A \in \mathfrak{R}$ *simple* if A and A^* do not have a common invariant subspace on which they coincide.

If A ($\in \mathfrak{R}$) is a nonsimple operator, then obviously there exists a *maximal* subspace $\mathfrak{L}_T = \mathfrak{L}_T(A)$, invariant with respect to A and A^*, on which these two operators coincide.

We shall call the subspace $\mathfrak{L}_T$ the *trivial* subspace of the nonselfadjoint operator A.

If $A = A^*$ then obviously $\mathfrak{L}_T(A) = \mathfrak{H}$. If A is a simple operator we put $\mathfrak{L}_T(A) = \{0\}$.

In the general case, as is easily seen, the subspace $\mathfrak{L}_T(A)$ consists of precisely those vectors $f \in \mathfrak{H}$ for which

$$A^n f = (A^*)^n f \qquad (n = 1, 2, \cdots). \tag{1.1}$$

From the definition of the trivial subspace it is clear that A and A^* have the same trivial subspace:

$$\mathfrak{L}_T(A) = \mathfrak{L}_T(A^*).$$

Obviously, the operators A and A^* can only be simple or nonsimple simultaneously.

1. *To every nonsimple operator $A \in \mathfrak{R}$ there corresponds a unique decomposition of $\mathfrak{H}$ into the orthogonal sum of two subspaces invariant with respect to A and A^*,*

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \tag{1.2}$$

having the following properties: 1) *A and A^* coincide on $\mathfrak{H}_1$, and* 2) *A induces a simple operator in $\mathfrak{H}_2$. In this decomposition, $\mathfrak{H}_1 = \mathfrak{L}_T(A)$.*

Indeed, $\mathfrak{L}_T = \mathfrak{L}_T(A)$ is contained in the subspace $\mathfrak{Z}_{A_{\mathscr{J}}}$ of all zeros of the operator $A_{\mathscr{J}}$ (the imaginary component of A). Therefore $A_{\mathscr{R}} = A$ on $\mathfrak{L}_T$, and so $\mathfrak{L}_T$ is invariant with respect to $A_{\mathscr{R}}$.

Since the operators $A_{\mathscr{R}}$ and $A_{\mathscr{J}}$ are selfadjoint, the orthogonal complement $\mathfrak{L}_T^{\perp}$ is also invariant with respect to these operators, and hence also with respect to the operators A and A^*. Obviously there are no vectors $f \neq 0$ in $\mathfrak{L}_T^{\perp}$ for which the condition (1.1) is fulfilled. Thus A induces a simple operator in $\mathfrak{L}_T^{\perp}$, and the decomposition

$$\mathfrak{H} = \mathfrak{L}_T \oplus \mathfrak{L}_T^{\perp}$$

has all the properties mentioned in 1.

We leave it to the reader to prove the uniqueness of a decomposition (1.2) with the indicated properties.

2. For a dissipative operator A the trivial subspace also has the following characteristic.

2. *The trivial subspace $\mathfrak{L}_T(A)$ of a dissipative operator coincides with the maximal invariant subspace of A in which A induces a selfadjoint operator.* [1)]

In fact, the operators A and A^* coincide on $\mathfrak{L}_T(A)$, so that A induces a selfadjoint operator in $\mathfrak{L}_T$. Let us now consider some invariant subspace $\mathfrak{L}_1$ of the operator A, in which A induces a selfadjoint operator. For $f \in \mathfrak{L}_1$ we have

$$(A_{\mathscr{I}} f, f) = \operatorname{Im}(Af, f) = 0,$$

and consequently the operator $A_{\mathscr{I}}$, being nonnegative, is identically zero on $\mathfrak{L}_1$. Thus $Af = A^*f$ for $f \in \mathfrak{L}_1$, and so $\mathfrak{L}_1 \subset \mathfrak{L}_T$, as was to be proved.

As a simple corollary of result 2 we obtain

3. *The trivial subspace of a completely continuous dissipative operator coincides with the closed linear hull of all the eigenvectors of the operator, corresponding to real eigenvalues (including $\lambda = 0$, if it is an eigenvalue).*

From result 3, in turn, follows

4. *A completely continuous simple dissipative operator induces in any one of its invariant subspaces an operator of the same type (completely continuous, simple and dissipative).*

For dissipative operators Lemma I.4.2 can be strengthened.

LEMMA 1.1. *Let A be a linear dissipative completely continuous operator, and $\mathfrak{E}$ the closed linear hull of all its root vectors. If the subspace $\mathfrak{E}^{\perp} = \mathfrak{H}_0$ consists of not only the zero vector, then the operator $P_0 A P_0$ is a simple dissipative Volterra operator in $\mathfrak{H}_0$.*

Here P_0 denotes the projector which projects $\mathfrak{H}$ orthogonally onto $\mathfrak{H}_0$.

PROOF. If we pass from consideration of the dissipative operator $A \in \mathfrak{S}_\infty$ to consideration of the simple dissipative operator $\hat{A}$, induced by A in $\mathfrak{L}_{\Pi} = \mathfrak{H} \ominus \mathfrak{L}_T(A)$, then by virtue of results 2 and 3

[1)] Results 1 and 2 can be generalized to unbounded maximal dissipative operators (see Langer [1] and Sz.-Nagy and Foiaş [1-4].

the space $\mathfrak{H}_0$ will not as a consequence be changed, nor will the operator $A_1 = P_0AP_0$ $(= P_0\hat{A}P_0)$. Therefore without loss of generality we may assume at once that A is a simple dissipative operator.

Then, recalling the notation of §4, Chapter I, we will have $\mathfrak{E} = \mathfrak{E}_A$, $\mathfrak{H}_0 = \mathfrak{M}$, $P_0 = Q_A$, and one can assert on the basis of Lemma I.4.2 that $A_1 = Q_AAQ_A$ is a Volterra operator. The adjoint A_1^* coincides on $\mathfrak{H}_0$ with the operator induced in $\mathfrak{H}_0$ by A^*.

Since by hypothesis the operator $A \in \mathfrak{S}_\infty$ is simple and dissipative, so is the operator A^*, hence also the operators A_1^* and A_1, considered in $\mathfrak{H}_0$. The lemma is proved.

3. We shall say that a property $(\mathscr{E})$, which is distinctive for certain bounded linear operators, is *invariant* with respect to orthogonal projection if it follows from an operator A having the property $(\mathscr{E})$ that the operator PAP has this property, where P is any orthogonal projector.

Thus for example the property of dissipativeness is a property invariant with respect to orthogonal projection.

LEMMA 1.2. *Let* $(\mathscr{K})$ *be some group of properties which are invariant with respect to orthogonal projection and distinctive for certain completely continuous dissipative operators. Then one of the following holds.*

1) *Every completely continuous dissipative operator having the group of properties* $(\mathscr{K})$ *has a system of root vectors which is complete in* $\mathfrak{H}$.

2) *There exists at least one simple dissipative Volterra operator which has the group of properties* $(\mathscr{K})$.

PROOF. In fact, suppose that 1) is not fulfilled, i.e. there exists some completely continuous dissipative operator A, having the group of properties $(\mathscr{K})$, such that the closed linear hull $\mathfrak{E}$ of all its root vectors forms a proper part of $\mathfrak{H}$. Let us put $\mathfrak{H}_0 = \mathfrak{E}^\perp$ and denote by P_0 the projector which orthogonally projects the space $\mathfrak{H}$ onto $\mathfrak{H}_0$. According to Lemma 1.1 the operator P_0AP_0 is a simple dissipative Volterra operator which, together with the operator A, has all the properties from the group $(\mathscr{K})$.

Thus if 1) is not fulfilled, then 2) is fulfilled.

On the other hand, it is obvious that if 2) is fulfilled, then 1) cannot hold. The lemma is proved.

This lemma admits a certain generalization to the case of nondissipative operators (compare V. B. Lidskiĭ [**5**], §1).

§2. Tests for the completeness of the system of root vectors for dissipative operators with nuclear imaginary component

1. The following general result holds.

THEOREM 2.1 (M. S. LIVŠIC [2]). *Let A be a simple dissipative or a completely continuous dissipative operator, with* $\operatorname{sp} A_{\mathcal{J}} < \infty$. *In order that the system of root vectors of A be complete, it is necessary and sufficient that*

$$\sum_{j=1}^{\nu(A)} \operatorname{Im} \lambda_j(A) = \operatorname{sp} A_{\mathcal{J}}. \tag{2.1}$$

PROOF. In fact, let $A\,(A_{\mathcal{J}} \in \mathfrak{S}_1)$ be an operator with a complete system of root vectors.[2] Using the method described in the proof of Lemma I.4.1, we form an orthonormal Schur basis $\{\omega_j\}_1^\infty$ from linear combinations of these root vectors. In this basis the operator A reduces to triangular form. Then

$$(A\omega_j, \omega_j) = \lambda_j(A) \quad \text{and} \quad (A_{\mathcal{J}}\omega_j, \omega_j) = \operatorname{Im} \lambda_j(A),$$

and so

$$\operatorname{sp} A_{\mathcal{J}} = \sum_{j=1}^{\infty} (A_{\mathcal{J}}\omega_j, \omega_j) = \sum_{j=1}^{\infty} \operatorname{Im} \lambda_j(A).$$

Conversely, suppose that (2.1) is fulfilled. We consider the closed linear hull $\mathfrak{E}_0$ of all the root vectors of the operator A, corresponding to its nonreal eigenvalues. Let us denote by $\hat{A}$ the dissipative operator induced by A in $\mathfrak{E}_0$, and form, in accordance with what was just proved, a Schur basis $\{\omega_j\}$ of the space $\mathfrak{E}_0$ such that

$$\sum_j (\hat{A}_{\mathcal{J}}\omega_j, \omega_j) = \sum_j \operatorname{Im} \lambda_j(\hat{A}) \left(= \sum_j \operatorname{Im} \lambda_j(A)\right).$$

Since

$$\sum_j (\hat{A}_{\mathcal{J}}\omega_j, \omega_j) = \sum_j \operatorname{Im}(\hat{A}\omega_j, \omega_j)$$
$$= \sum_j \operatorname{Im}(A\omega_j, \omega_j) = \sum_j (A_{\mathcal{J}}\omega_j, \omega_j),$$

it follows from (2.1) that

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[2] In proving the necessity of the condition (2.1) the dissipativeness or complete continuity of A is not used.

$$\sum_j (A_{\mathscr{J}}\omega_j, \omega_j) = \operatorname{sp} A_{\mathscr{J}}.$$

Since the operator $A_{\mathscr{J}}$ is nonnegative, we conclude from the last equality that it is identically zero on the subspace $\mathfrak{H}_0 = \mathfrak{H} \ominus \mathfrak{E}_0$. Thus on $\mathfrak{H}_0$ we have $A = A_{\mathscr{R}} = A^*$ and consequently A is selfadjoint. Therefore if A is a simple operator $\mathfrak{H}_0$ consists only of zero; if A is completely continuous, then it has a complete system of eigenvectors in $\mathfrak{H}_0$. Thus in both cases the theorem is proved.

M. S. Livšic obtained his theorem as a corollary of a rather complicated construction, related to the triangular model which he found for bounded linear operators A with $A_{\mathscr{J}} \in \mathfrak{S}_1$. The elementary proof presented here, whose concept borders upon that of the proof by I. Schur of his inequality (cf. Theorem 6.1), is due to B. R. Mukminov [1]. [3)]

When applied to a Fredholm integral operator, the theorem gives the following result.

1. *Let* $\mathscr{A}(t,s)$ $(a \leqq t, s \leqq b)$ *be a Hilbert-Schmidt kernel, for which the kernel*

$$\mathscr{A}_{\mathscr{J}}(t,s) = (\mathscr{A}(t,s) - \overline{\mathscr{A}(s,t)})/2i$$

is Hermitian nonnegative and satisfies the condition

$$S = \overline{\lim_{h\to 0}} \frac{1}{4h^2} \int_a^b \int_a^b [2h - |t-s|\,]_+ \mathscr{A}_{\mathscr{J}}(t,s)\, ds dt < \infty. \tag{2.2}$$

Then in order that the system of root vectors of the operator

$$(Af)(t) = \int_a^b \mathscr{A}(t,s) f(s)\, ds$$

be complete in $L_2(a,b)$, *it is necessary and sufficient that*

$$\sum_j \operatorname{Im} \lambda_j(A) = S.$$

This obviously follows from Theorems III.10.1 and 2.1. The condition (2.2) is always fulfilled if $\mathscr{A}_{\mathscr{J}}(t,s)$ is a bounded kernel. [4)]

[3)] It also enables one to generalize Livšic's theorem to unbounded operators (cf. Gohberg and Kreĭn [1]).

[4)] For the case of a bounded kernel $\mathscr{A}(t,s)$ the corresponding result was formulated in a paper by Livšic [2], in which the quantity S was defined by $S = \int_a^b \mathscr{A}_{\mathscr{J}}(t,t)\,dt$, in which the right side, for the case of a discontinuous kernel $\mathscr{A}(t,s)$, has no relation (generally speaking) to the trace of the operator $A_{\mathscr{J}}$. This oversight carried over to other papers (Brodskiĭ and Livšic [1], Mukminov [1]).

Let us further recall that if the limit supremum in (2.2) is finite, then the ordinary limit exists and coincides with it (see p. 115).

If the kernel $\mathscr{A}(t,s)$ is continuous, then

$$S = \int_a^b \mathscr{A}_{\mathscr{J}}(t,t)\,dt.$$

2. Let us agree to say that the operator A can be broken up into the *orthogonal direct sum* (*difference*) of the operators A_1 and A_2, and to write

$$A = A_1 \oplus A_2 \qquad (A = A_1 \ominus A_2),$$

if the space $\mathfrak{H}$ can be broken up into the orthogonal sum of two subspaces invariant with respect to A,

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2,$$

such that the operator A induces in $\mathfrak{H}_1$ the operator A_1, and in $\mathfrak{H}_2$ the operator A_2 $(-A_2)$. It was shown in §1.2 that every bounded dissipative operator can be broken up into the orthogonal direct sum of a selfadjoint operator and a simple dissipative operator.

The theorem of M. S. Livšic admits the following generalization.

Theorem 2.2. *Let A be a simple or completely continuous operator and suppose that $A_{\mathscr{J}} \in \mathfrak{S}_1$. In order that the operator A satisfy the condition*

$$\sum_{j=1}^{\nu(A)} |\operatorname{Im}\lambda_j(A)| = |A_{\mathscr{J}}|_1 \left(= \sum_j |\lambda_j(A_{\mathscr{J}})| \right), \tag{2.3}$$

it is necessary and sufficient that

a) *it have a complete system of root vectors, and*

b) *it can be broken up into the orthogonal direct difference of two dissipative operators.*

Proof. We shall first prove the sufficiency of conditions a) and b). Let us denote by A_1 and A_2 the dissipative operators into whose orthogonal direct difference A can be broken up: $A = A_1 \ominus A_2$. If the operator A has the property a), then A_1 and A_2 have the same property. Hence, by Theorem 2.1,

$$\sum_j \operatorname{Im}\lambda_j(A_k) = \operatorname{sp}(A_k)_{\mathscr{J}} = |(A_k)_{\mathscr{J}}|_1 \qquad (k = 1, 2). \tag{2.4}$$

The complete system of eigenvalues of the operator A consists of the

union of the sequences

$$\{\lambda_j(A_1)\} \quad \text{and} \quad \{-\lambda_j(A_2)\},$$

and moreover

$$|A_{\mathscr{J}}|_1 = |(A_1)_{\mathscr{J}}|_1 + |(A_2)_{\mathscr{J}}|_1.$$

By (2.4) it follows that (2.3) is fulfilled for the operator A.

Conversely, suppose that (2.3) holds for A. We form the closed linear hull $\mathfrak{E}_A$ of all the root vectors of A corresponding to nonzero eigenvalues. Let $\{\omega_j\}$ be the basis of $\mathfrak{E}_A$ whose existence was asserted in Lemma I.4.1. Then from the equalities

$$(A_{\mathscr{J}}\omega_j, \omega_j) = \operatorname{Im}\lambda_j(A) \qquad (j = 1, 2, \cdots)$$

it follows that

$$\sum_j |(A_{\mathscr{J}}\omega_j, \omega_j)| = |A_{\mathscr{J}}|_1.$$

By Theorem III.8.6 this means that the operator $UA_{\mathscr{J}}$ is nonnegative and vanishes on the subspace $\mathfrak{H}_0 = \mathfrak{E}_A^{\perp}$, where U is the unitary operator for which $Uf = f$ for $f \in \mathfrak{E}_A^{\perp}$ and

$$U\omega_j = \epsilon_j\omega_j,$$

where

$$\epsilon_j = \operatorname{sign}\operatorname{Im}\lambda_j(A) \quad \text{for} \quad \operatorname{Im}\lambda_j(A) \neq 0,$$

$$\epsilon_j = 1 \qquad\qquad \text{for} \quad \operatorname{Im}\lambda_j(A) = 0.$$

It is obvious that each of the subspaces $\mathfrak{H}_0$ and $\mathfrak{E}_A$ is invariant with respect to the operators $A_{\mathscr{J}}$, $A_{\mathscr{R}}$, A and A^*. Since $A_{\mathscr{J}}\mathfrak{H}_0 = 0$, the operator A is selfadjoint on the subspace $\mathfrak{H}_0$. Moreover, according to Lemma I.4.2 the operator A does not have eigenvalues different from zero in this subspace. Consequently

$$A\mathfrak{H}_0 = A_{\mathscr{R}}\mathfrak{H}_0 = 0.$$

From the foregoing it follows, in particular, that the operator A has a complete system of root vectors. We shall now consider the operators U, A, $A_{\mathscr{R}}$, $A_{\mathscr{J}}$ as operators acting in $\mathfrak{E}_A$. The selfadjointness of $UA_{\mathscr{J}}$ implies that the operators U and $A_{\mathscr{J}}$ commute.

In the basis $\{\omega_j\}$ there corresponds to the operator A a triangular matrix

$$\| (A\omega_k, \omega_j) \|_{j,k} = \| a_{jk} \|_{j,k} \qquad (a_{jk} = 0 \text{ for } j > k).$$

In the same basis there corresponds to $A_{\mathscr{J}}$ the matrix $\| h_{jk} \|$ with elements

$$(2.5) \qquad h_{jk} = a_{jk}/2i \quad \text{for} \quad j < k \quad \text{and} \quad h_{jk} = -\bar{a}_{kj}/2i \quad \text{for} \quad j > k.$$

Since to the operators $UA_{\mathscr{J}}$ and $A_{\mathscr{J}}U$ there correspond, in the basis being considered, the matrices $\| \epsilon_j h_{jk} \|$ and $\| h_{jk} \epsilon_k \|$ respectively, the commutativity of the operator U and $A_{\mathscr{J}}$ is equivalent to

$$(2.6) \qquad \epsilon_j h_{jk} \epsilon_k = h_{jk},$$

which shows that $h_{jk} = 0$ for all j, k for which $\epsilon_j \neq \epsilon_k$.

From (2.5) and (2.6) it follows that

$$\epsilon_j a_{jk} \epsilon_k = a_{jk};$$

hence the operators A and U also commute: $AU = UA$. Let us denote by P and Q the orthogonal projectors defined by

$$P = (I + U)/2 \quad \text{and} \quad Q = (I - U)/2.$$

Obviously $PQ = QP = 0$ and $P + Q = I$. It is also obvious that each of the operators A, $A_{\mathscr{R}}$ and $A_{\mathscr{J}}$ commutes with each of the orthoprojectors P and Q.

The operator A can be broken up into the orthogonal direct difference of two operators:

$$A = A_1 \ominus A_2,$$

where

$$A_1 = PAP = PA_{\mathscr{R}}P + iPA_{\mathscr{J}}P \quad \text{and} \quad -A_2 = QAQ = QA_{\mathscr{R}}Q + iQA_{\mathscr{J}}Q.$$

Since the operator $UA_{\mathscr{J}}$ is nonnegative, so are the operators $PA_{\mathscr{J}}P$ and $-QA_{\mathscr{J}}Q$, since

$$UA_{\mathscr{J}} = (P - Q)A_{\mathscr{J}} = PA_{\mathscr{J}}P - QA_{\mathscr{J}}Q.$$

Consequently the operators A_1 and A_2 are dissipative. The theorem is proved.

3. The tests for the completeness in $\mathfrak{H}$ of a system of root vectors, formulated in Theorems 2.1 and 2.2, have the peculiarity that, generally speaking, their verification is possible only after the spectrum of the operator has been computed. However, from them one can obtain simple sufficient conditions; for example, a comparison of Theo-

rem 2.1 with Theorem III.8.4 of V. B. Lidskiĭ leads to a test which does not require the calculation of the spectrum.

THEOREM 2.3 (LIDSKIĬ [5,6]). *If the dissipative operator A belongs to $\mathfrak{S}_1$, then its system of root vectors is complete in $\mathfrak{H}$.*

PROOF. In fact, by Theorem III.8.4

$$\sum_j \lambda_j(A) = \operatorname{sp} A = \operatorname{sp} A_{\mathscr{R}} + i \operatorname{sp} A_{\mathscr{J}}.$$

Consequently

$$\sum_j \operatorname{Im} \lambda_j(A) = \operatorname{sp} A_{\mathscr{J}}.$$

The theorem is proved.

Other stronger versions of this theorem are discussed in §§4—6.

§3. Tests for the completeness of the system of root vectors of a contraction operator[5)]

An operator $A \in \mathfrak{R}$ is called a *contraction operator* or simply a *contraction,* if $|A| \leqq 1$.

The theory of dissipative operators is closely connected with the theory of contractions. It is easily seen that if B is a dissipative operator, then its Cayley transform

$$A = (B - iI)(B + iI)^{-1}$$

is a contraction. This situation can be extended to so-called unbounded maximal dissipative operators and plays a fundamental role in their theory (cf. Phillips [1], Sz.-Nagy and Foiaş [1—4], Kato [1], Langer [1,2]). However we shall not touch upon any of this.

The basic aim of this section is to establish a theorem on the completeness of the system of root vectors of a contraction, which should be considered as a certain analog (of course, having a more definitive character) of the corresponding theorem of M. S. Livsič for dissipative operators (Theorem 2.1).

1. First we shall show that *if the operator $A \in \mathfrak{R}$ is invertible, and the operator $H_1 = A^*A - I$ belongs to some two-sided ideal $\mathfrak{S}$, then the polar representation of the operator A has the form*

$$A = U(I + H), \tag{3.1}$$

5) The results of this section are taken over from the authors' paper [6].

where U is a unitary operator, and $H \in \mathfrak{S}$.

In fact, according to §1.4, Chapter I, the operator A has the form (3.1), where $I + H$ is a nonnegative operator such that $(I+H)^2 = I + H_1$. It follows that $H \in \mathfrak{S}_\infty$ and

$$\lambda_n(H) = \sqrt{1 + \lambda_n(H_1)} - 1 = \frac{1}{2}\lambda_n(H_1) + O(\lambda_n^2(H_1)) \qquad (n \to \infty).$$

Hence $\lambda_n(H) = O(\lambda_n(H_1))$ $(n \to \infty)$ and so H, along with H_1, belongs to $\mathfrak{S}$.

According to Theorem I.5.3, for every invertible operator A of the form (3.1), that part of its spectrum which does not lie on the unit circle consists of normal eigenvalues. We now enumerate, in any way, all the eigenvalues not lying on the unit circle, counting each one as many times as its algebraic multiplicity. We shall in this section denote the sequence of these eigenvalues by $\{\lambda_j'(A)\}$.

LEMMA 3.1. *Let A be an invertible operator for which $A^*A - I \in \mathfrak{S}_1$. If $\mathfrak{E}$ is the closed linear hull of all the root vectors of the operator A corresponding to the eigenvalues $\lambda_j'(A)$, and P is the orthoprojector which projects the space $\mathfrak{H}$ onto $\mathfrak{E}$, then*

$$\det(PA^*AP + Q) = \prod_j |\lambda_j'(A)|^2, \tag{3.2}$$

where $Q = I - P$. In particular, if the indicated set of root vectors is dense in $\mathfrak{H}$, i.e., $\mathfrak{E} = \mathfrak{H}$, then

$$\det A^*A = \prod_j |\lambda_j'(A)|^2. \tag{3.3}$$

PROOF. It is easily seen that (3.3) will be proved as soon as (3.2) is proved. For the proof of the latter we construct (as in Lemma I.4.1) an orthonormal Schur basis $\{\omega_j\}_1^\infty$ for which the matrix corresponding to the operator A is triangular, i.e.

$$A\omega_j = a_{j1}\omega_1 + \cdots + a_{jj-1}\omega_{j-1} + \lambda_j(A)\,\omega_j \qquad (j = 1, 2, \cdots).$$

Let us denote by P_n the orthoprojector which projects $\mathfrak{H}$ onto the subspace with basis $\{\omega_j\}_1^n$. Obviously $AP_n = P_nAP_n$ and consequently $P_nA^*P_nAP_n = P_nA^*AP_n$. Since

$$\det(A^*A) = \lim_{n\to\infty} \det(P_nA^*AP_n + Q_n)$$
$$= \lim_{n\to\infty} \det(P_nA^*P_nAP_n + Q_n),$$

and

$$\det(P_nA^*P_nAP_n + Q_n)$$
$$= \det(P_nA^*P_n + Q_n)\det(P_nAP_n + Q_n) = \prod_{j=1}^{n} |\lambda_j'(A)|^2,$$

we obtain (3.2).

For what follows, we shall need the following general lemma.

LEMMA 3.2. *Let $A \in \mathfrak{R}$ be an arbitrary invertible operator for which $A^*A - I \in \mathfrak{S}_1$. If $\mathfrak{L}$ is an invariant subspace of each of the operators A and A^{-1}, then*

$$\det(A^*A) = \det(PA^*AP + Q)\det(QA^*QAQ + P), \tag{3.4}$$

where P is the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{L}$, and $Q = I - P$.

PROOF. Since $QAP = QPAP = 0$, we can represent the operator A in the form

$$A = (P + Q)A(P + Q) = PAP + PAQ + QAQ.$$

It follows that

$$A = (QAQ + P)(I + PAQ)(Q + PAP). \tag{3.5}$$

The operator $S_1 = Q + PAP$ has a bounded inverse

$$S_1^{-1} = Q + PA^{-1}P.$$

The square of the operator PAQ equals zero, and consequently the operator $S_2 = I + PAQ$ is invertible. But then we conclude from (3.5) that $S_3 = QAQ + P$ is also invertible.

Since the operator

$$S_1^*S_1 - I = PA^*PAP + Q - I = P(A^*A - I)P$$

is nuclear, the operator

$$S_1S_1^* - I = (S_1^*)^{-1}(S_1^*S_1 - I)S_1^*$$

is nuclear. From the last relation, according to result 6, §1, Chapter IV, it also follows that

$$\det(S_1S_1^*) = \det(S_1^*S_1).$$

Further, taking into account the equality

$$S_1A^*AS_1^{-1} = S_1S_1^*S_2^*S_3^*S_3S_2 \qquad (= S_1S_1^*B),$$

we find that $B - I \in \mathfrak{S}_1$ and

$$\det(A^*A) = \det(S_1^*S_1)\det B. \tag{3.6}$$

Similarly, we deduce from the equality

$$S_2BS_2^{-1} = S_2S_2^*S_3^*S_3$$

that $S_2S_2^*S_3^*S_3 - I \in \mathfrak{S}_1$ and

$$\det B = \det(S_2S_2^*S_3^*S_3). \tag{3.7}$$

We note that $S_3S_3^* - I = P + QAQA^*Q - I = Q(AA^* - I)Q$.

Since $AA^* - I = A^{*-1}(A^*A - I)A^* \in \mathfrak{S}_1$, the operator $S_3S_3^* - I$ also belongs to $\mathfrak{S}_1$. Finally, from the equality $S_3^*S_3 - I = S_3^{-1}(S_3S_3^* - I)S_3$ we obtain $S_3^*S_3 - I \in \mathfrak{S}_1$ and

$$\det(S_3^*S_3) = \det(S_3S_3^*). \tag{3.8}$$

From what has been proved it follows that the operator $S_2S_2^* - I = PAQ + QA^*P + PAQA^*P$ is nuclear; consequently the operator $S_2 - I = PAQ = P(S_2S_2^* - I)Q$ is nuclear.

Since PAQ is a Volterra operator, we have

$$\det S_2 = \det S_2^* = 1. \tag{3.9}$$

Comparing (3.6)—(3.9), we arrive at (3.4). The lemma is proved.

2. We now proceed to the consideration of contractions.

Obviously, the entire spectrum of a contraction A ($|A| \leqq 1$) is contained in the closed unit disc.

1. *Let A ($\in \mathfrak{R}$) be a contraction. If for some orthoprojector P the operator PAP is unitary in the subspace $P\mathfrak{H}$, i.e.,*

$$PA^*PAP = PAPA^*P = P,$$

then each of the subspaces $P\mathfrak{H}$ and $(I - P)\mathfrak{H}$ is invariant with respect to A.

In fact,

$$PA^*AP = PA^*PAP + PA^*QAP = P + PA^*QAP, \tag{3.10}$$

where $Q = I - P$. Since the operator PA^*AP is a contraction, and the operator $PA^*QAP = (QAP)^*(QAP)$ is nonnegative, the equality (3.10) can hold if and only if this operator equals zero, i.e. $QAP = 0$. This means that the subspace $P\mathfrak{H}$ is invariant with respect to A. Similarly, from the equality

$$PAA^*P = PAPA^*P + PAQA^*P = P + PAQA^*P$$

we deduce that $PAQ = 0$, and consequently the subspace $Q\mathfrak{H}$ is invariant with respect to A.

We shall say that a contraction $A \in \mathfrak{R}$ is *simple*, if it does not induce a unitary operator in any one of its invariant subspaces.

It follows from result 1 that if A is a simple contraction, the operator PAP will not be unitary in the subspace $P\mathfrak{H}$ for any orthoprojector P.

THEOREM 3.1 (LANGER [1]; SZ.-NAGY AND FOIAŞ [1]). *Let $A \in \mathfrak{R}$ be a contraction. Then the set $\mathfrak{H}_A$ of all vectors $f \in \mathfrak{H}$ for which*

$$|A^n f| = |A^{*n} f| = |f| \qquad (n = 1, 2, \cdots)$$

forms the maximal invariant subspace of the operator A in which it induces a unitary operator. The subspace $\mathfrak{H}_A^{\perp}$ is also invariant with respect to A, and A induces a simple contraction in it.

Thus every contraction can be broken up into the orthogonal direct sum of a unitary operator and a simple contraction.

PROOF. In fact, if $f \in \mathfrak{H}_A$ then for any positive integer n

$$|A^{*n}A^n f - f|^2 = |A^{*n}A^n f|^2 - 2|A^n f|^2 + |f|^2 \leqq |f|^2 - |A^n f|^2 = 0,$$

i.e.

$$A^{*n}A^n f = f \qquad (f \in \mathfrak{H}_A;\, n = 1, 2, \cdots). \tag{3.11}$$

Interchanging the roles of A and A^*, we obtain

$$A^n A^{*n} f = f \qquad (f \in \mathfrak{H}_A;\, n = 1, 2, \cdots). \tag{3.12}$$

It is easily seen that the converse is true; if for some vector $f \in \mathfrak{H}$ the relations (3.11) and (3.12) hold, then $f \in \mathfrak{H}_A$. Thus $\mathfrak{H}_A$ consists of precisely those vectors $f \in \mathfrak{H}$ for which the relations (3.11) and (3.12) hold. It follows at once that $\mathfrak{H}_A$ is a closed subspace of $\mathfrak{H}$.

For any vector $g = Af$ $(f \in \mathfrak{H}_A)$ we have

$$|A^n g| = |A^{n+1} f| = |Af| = |g|$$

and

$$|A^{*n} g| = |A^{*n-1}A^*Af| = |A^{*n-1}f| = |Af| = |g|.$$

Consequently $\mathfrak{H}_A$ is invariant with respect to A. One deduces similarly that $\mathfrak{H}_A$ is invariant with respect to A^*.

By virtue of (3.11) and (3.12) for $n = 1$, the operator A induces a unitary operator in $\mathfrak{H}_A$. If some subspace $\mathfrak{L}$ is invariant with respect

to A, and A induces a unitary operator in $\mathfrak{L}$, then according to result 1 this subspace is invariant with respect to A^*, and so $\mathfrak{L} \subset \mathfrak{H}_A$.

The subspace $\mathfrak{H}_A^\perp$ is likewise invariant with respect to the operators A and A^*, and for any $f \in \mathfrak{H}_A^\perp$ with $|f| = 1$ there is a positive integer n such that at least one of the numbers $|A^n f|$, $|A^{*n} f|$ is less than unity.

Consequently the operator A induces a simple contraction in $\mathfrak{H}_A^\perp$.

THEOREM 3.2. *Let $A \in \mathfrak{R}$ be any contraction for which $A^{-1} \in \mathfrak{R}$ and $A^*A - I \in \mathfrak{S}_1$. Then*

$$\det(A^*A) \leqq \prod_j |\lambda_j(A)|^2. \tag{3.13}$$

If, moreover, the operator A is simple or differs from the identity operator by a completely continuous operator, i.e. $A - I \in \mathfrak{S}_\infty$, then the equal sign in (3.13) *holds if and only if the system of all root vectors of the operator A is complete in $\mathfrak{H}$.*

PROOF. Let $\mathfrak{E}$ be the closed linear hull of all the root vectors of the operator A corresponding to the eigenvalues $\lambda_j(A)$, and let P be the orthoprojector which projects the space $\mathfrak{H}$ onto $\mathfrak{E}$. Obviously $\mathfrak{E}$ is an invariant subspace of the operators A and A^{-1}. Consequently, by Lemma 3.2,

$$\det(A^*A) = \det(PA^*AP + Q)\det(P + QA^*QAQ). \tag{3.14}$$

The operator

$$P + QA^*QAQ = I - (Q - QA^*QAQ) \leqq I$$

is nonnegative, and so

$$\det(P + QA^*QAQ) \leqq 1. \tag{3.15}$$

Taking into consideration (3.14), (3.15) and (3.2), we arrive at the relation (3.13).

The equal sign holds in (3.13) if and only if it holds in (3.15), and this is the case if and only if

$$QA^*QAQ = Q. \tag{3.16}$$

In this case

$$QA^*AQ = QA^*QAQ + QA^*PAQ = Q + QA^*PAQ,$$

from which it follows that $PAQ = 0$ and hence that $Q\mathfrak{H}$ is an invariant subspace of the operator A. Since, moreover, $P\mathfrak{H}$ is invariant

with respect to A and A is invertible, A can be broken up into the orthogonal direct sum of two invertible operators. By virtue of (3.16) the operator corresponding to the subspace $Q\mathfrak{H}$ is unitary.

If A is a simple contraction then $Q = 0$, and so $\mathfrak{E} = \mathfrak{H}$. If $A - I \in \mathfrak{S}_\infty$, then the unitary restriction of the operator A to the subspace $Q\mathfrak{H}$ has a complete system of eigenvectors in $Q\mathfrak{H}$. The theorem is proved.

A theorem close to (but with a more complicated statement than) Theorem 3.2 was established by V. G. Poljackiĭ [1]. In deriving his result Poljackiĭ used the triangular models of "quasi-unitary" operators which he had obtained. In constructing these models he drew upon the deep investigations of V. P. Potapov [1] and Ju. P. Ginzburg [1] in the theory of analytic matrix and operator functions.

REMARK 3.1. A theorem similar to Theorem 3.2 can be formulated for dilations, i.e. for operators $A \in \mathfrak{R}$ having the property that $|Af| \geqq |f|$. In this case inequality (3.13) goes over into the inequality

$$\det(A^*A) \geqq \prod_j |\lambda_j'(A)|^2,$$

where $\{\lambda_j'(A)\}$ denotes the complete system of eigenvalues of the operator A lying outside the unit circle.

If A is an invertible dilation, then $B = A^{-1}$ will be an invertible contraction, and conversely. Therefore each of these theorems leads to the other.

§4. Theorems on tests for the completeness of the systems of root vectors of dissipative operators with nuclear imaginary component[6)]

In this section we shall present three theorems on the completeness of the system of root vectors of a dissipative operator with nuclear imaginary component. The first two theorems contain Theorem 3.2 of V. B. Lidskiĭ as a corollary.

The first of them guarantees the completeness of the system of root vectors of an operator A of the indicated type so long as the real component $A_{\mathcal{R}}$ has a "sufficiently sparse" spectrum on the positive or negative semi-axis, and the second—when the corresponding condition is fulfilled for the s-numbers of this operator.

The third theorem shows that the completeness of the system of

[6)] The results of this section are due to M. G. Kreĭn and are essentially taken over from his paper [9].

root vectors holds if and only if the spectra of the operators A and $A_{\mathscr{R}}$ have "distribution densities" of the same order.

1. The following theorem is easily deduced from Theorem IV.8.2.

THEOREM 4.1 (M. G. KREĬN [9]). *The system of root vectors of a completely continuous dissipative operator A with nuclear imaginary component is complete in $\mathfrak{H}$ so long as at least one of the following two conditions is fulfilled*:

$$\lim_{\rho\to\infty}\frac{n_+(\rho;A_{\mathscr{R}})}{\rho}=0 \quad \textit{or} \quad \lim_{\rho\to\infty}\frac{n_-(\rho;A_{\mathscr{R}})}{\rho}=0. \tag{4.1}$$

We recall that $n_\pm(\rho;A_{\mathscr{R}})$ denotes the number of characteristic numbers of the real component $A_{\mathscr{R}}$ in the interval $[0,\rho]$ and $[-\rho,0]$, respectively.

PROOF. The properties stated in the hypothesis of the theorem are invariant with respect to the operation of orthogonal projection. This requires clarification only for the conditions in (4.1). If P is an orthogonal projector and $\hat{A}=PAP$, then $\hat{A}_{\mathscr{R}}=PA_{\mathscr{R}}P$, and on the basis of the minimax properties of the eigenvalues of a selfadjoint operator we can assert [7] that $\lambda_j^+(\hat{A}_{\mathscr{R}})\leqq\lambda_j^+(A_{\mathscr{R}})$ and $\lambda_j^-(\hat{A}_{\mathscr{R}})\geqq\lambda_j^-(A_{\mathscr{R}})$ $(j=1,2,\cdots)$ (for this it is sufficient to represent the operator $A_{\mathscr{R}}$ as the difference of its nonnegative components and to use Lemma II.1.2). It follows that

$$n_\pm(\rho;\hat{A}_{\mathscr{R}})\leqq n_\pm(\rho;A_{\mathscr{R}}) \qquad (0<\rho<\infty).$$

Therefore if $A_{\mathscr{R}}$ has one of the properties (4.1), $\hat{A}_{\mathscr{R}}$ will have the same property.

On the basis of Lemma 1.2, to complete the proof it remains to show that no dissipative Volterra operator A satisfying the condition $0<\operatorname{sp}A_{\mathscr{J}}<\infty$ can have one of the properties (4.1). For this we note that by Theorem IV.7.2 we always have, for such an operator,

$$\lim_{\rho\to\infty}\frac{n_\pm(\rho;A_{\mathscr{R}})}{\rho}=\frac{1}{\pi}\operatorname{sp}A_{\mathscr{J}}. \tag{4.2}$$

The theorem is proved.

REMARK 4.1. Obviously the operator $A_{\mathscr{R}}$ will satisfy one of the conditions (4.1) whenever it has only a finite number of eigenvalues

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[7] Here we denote by $\lambda_j^+(A)$ $(\lambda_j^-(A))$ $(j=1,2,\cdots)$ the sequence in decreasing (increasing) order of the positive (negative) eigenvalues of the operator A.

of the corresponding sign. More generally, the first (second) of the conditions (4.1) will certainly be fulfilled whenever the sum

$$\sum_{\lambda_n>0} \lambda_n(A_{\mathscr{R}}) \qquad \left(\sum_{\lambda_n<0} \lambda_n(A_{\mathscr{R}})\right)$$

is finite.

The following theorem is close to Theorem 4.1 in content and, in particular, in method of proof.

THEOREM 4.2. *The system of root vectors of a dissipative operator A with nuclear imaginary component is complete whenever*

$$\varliminf_{n\to\infty} ns_n(A) = 0. \tag{4.3}$$

PROOF. Since for any orthoprojector P we have

$$s_n(PAP) \leqq s_n(A) \qquad (n = 1, 2, \cdots),$$

the property (4.3) is invariant with respect to the operation of orthogonal projection.

On the other hand, if A is a dissipative Volterra operator with $\operatorname{sp} A_{\mathscr{J}} > 0$, then according to Corollary IV.7.2

$$\lim_{n\to\infty} ns_n(A) = \frac{2}{\pi} \operatorname{sp} A_{\mathscr{J}}.$$

By virtue of Lemma 1.2, the theorem is proved.

2. Analysis of the "accuracy" of the conditions of Theorem 4.1.

a) The theory of the abstract triangular representation (see Brodskiĭ, Gohberg, Kreĭn and Macaev [1]) enables us to show that, given any selfadjoint operator $H \in \mathfrak{S}_1$, we can construct in uncountably many ways a Volterra operator A with $A_{\mathscr{J}} = H$, and if $H \geqq 0$, then A will satisfy the relation (4.2).

This circumstance indicates that if we retain the condition $\operatorname{sp} A_{\mathscr{J}} < \infty$ in Theorem 4.1, *no* additional restriction on the *dimension* of the operator $A_{\mathscr{J}}$ or on the *behavior of its spectrum* permits a weakening of the condition (4.1).

In passing, we recall that in §7 of Chapter IV we considered the operator J of integration in $L_2(0,1)$, an example of a dissipative Volterra operator with a one-dimensional imaginary component.

b) On the other hand, if we discard in Theorem 4.1 the requirement that the operator A be dissipative (more generally, that $A_{\mathscr{J}} \leqq 0$ or $\geqq 0$), then the theorem ceases to be valid.

Moreover, it turns out that there exist *Volterra* operators V with a *two-dimensional* imaginary component $V_{\mathscr{I}}$, and a real component $V_{\mathscr{R}}$ having only one negative characteristic number and an *arbitrarily sparse* spectrum of positive characteristic numbers (*sparse* in the sense of small exponent of convergence).

In fact, let $\sigma(t)$ $(0 \leqq t \leqq 1; \sigma(0) = 0)$ be a nondecreasing function with an infinite number of points of increase, and let the Hilbert space $\mathfrak{H}$ be realized as the space $L_{2,\sigma}$ of σ-measurable functions $f(t)$ $(0 \leqq t \leqq 1)$ with σ-integrable squares. The scalar product in $L_{2,\sigma}$ is of course defined by

$$(f,g) = \int_0^1 f(t)\,\overline{g(t)}\,d\sigma(t).$$

Let us consider the Volterra operator V[8] in $L_{2,\sigma}$ defined by

$$(Vf)(t) = -\int_0^t (t-s)\,f(s)\,d\sigma(s), \tag{4.4}$$

for which, as is easily seen,

$$\begin{aligned}(V_{\mathscr{R}}f)(t) &= -\frac{1}{2}\int_0^1 |t-s|\,f(s)\,d\sigma(s),\\ (V_{\mathscr{I}}f)(t) &= -\frac{1}{2i}\int_0^1 (t-s)\,f(s)\,d\sigma(s).\end{aligned} \tag{4.5}$$

It is obvious from (4.5) that the component $V_{\mathscr{I}}$ is two-dimensional and that its trace is zero.

The component $V_{\mathscr{R}}$ also has zero trace and therefore has eigenvalues of both signs. We shall show that $V_{\mathscr{R}}$ has exactly one negative eigenvalue. To do this, we consider the selfadjoint operator Γ in $L_{2,\sigma}$ defined by

$$(\Gamma f)(t) = \int_0^1 G(t,s)\,f(s)\,d\sigma(s),$$

where

$$G(t,s) = \begin{cases} t(1-s), & t \leqq s,\\ s(1-t), & s \leqq t,\end{cases}$$

is the well-known kernel from the theory of string motion (the func-

[8] For $\sigma(t) = t$ this example of a Volterra operator was considered in §10, Chapter IV.

tion $G(t,s)$ coincides, up to a scalar factor, with the Green's function of a homogeneous string).

Since $G(t,s)$ is a positive definite kernel, the operator Γ is positive. On the other hand, since

$$-\tfrac{1}{2}|t-s| = G(t,s) + (t-\tfrac{1}{2})(s-\tfrac{1}{2}) - \tfrac{1}{4},$$

we see that

$$V_{\mathscr{R}} = \Gamma + (\cdot, e_1)\, e_1 - (\cdot, e_0)\, e_0, \tag{4.6}$$

where $e_1(t) = t - \frac{1}{2}$, $e_0(t) = \frac{1}{2}$. Thus the operator $V_{\mathscr{R}}$ is obtained from the positive operator $\Gamma + (\cdot, e_1)e_1$ by subtracting a one-dimensional operator from it, and consequently has not more that one negative eigenvalue (cf. Lemma II.1.2), and since $\operatorname{sp} V_{\mathscr{R}} = 0$, exactly one.

The component $V_{\mathscr{R}}$ satisfies both of the conditions (4.1). Moreover, it is obvious from the relation (4.6) that the characteristic numbers of the operator $V_{\mathscr{R}}$ have the same asymptotic behavior as those of the operator Γ, in consequence of which (see M. G. Kreĭn [4] and the authors' book [7])

$$\begin{aligned}\lim_{\rho\to\infty}\left[\frac{1}{\sqrt{\rho}}\, n(\rho; V_{\mathscr{R}})\right] &= \lim_{\rho\to\infty}\left[\frac{1}{\sqrt{\rho}}\, n_+(\rho; V_{\mathscr{R}})\right] \\ &= \frac{1}{\pi}\int_0^1 \sqrt{\sigma'(t)}\, dt.\end{aligned} \tag{4.7}$$

(The derivative $\sigma'(t)$ of the nondecreasing function $\sigma(t)$ exists almost everywhere.)

The limit in (4.7) will equal zero if and only if the function $\sigma(t)$ lacks an absolutely continuous part. In particular, the limit in (4.7) equals zero in the *Stieltjes case*, i.e., when $\sigma(t)$ is a pure jump function whose jump points form a sequence tending to unity.

The theory of inverse problems for a string (see M. G. Kreĭn [6, 7]) enables us to assert that for any $\kappa \leqq 2$ one can choose a pure jump function $\sigma(t)$ (of Stieltjes type) such that

$$\lim_{\rho\to\infty} \frac{n_+(\rho; V_{\mathscr{R}})}{\rho^{\kappa}} = \lim_{\rho\to\infty} \frac{n_+(\rho; \Gamma)}{\rho^{\kappa}} = 1.$$

Thus by a suitable choice of the distribution function $\sigma(t)$ one can obtain any sparseness of the spectrum of the real component $V_{\mathscr{R}}$.

At the same time, for any choice of $\sigma(t)$ *both components* $V_{\mathscr{R}}$ *and* $V_{\mathscr{J}}$ *of the Volterra operator* $V = V^{(\sigma)}$ *have* (*each*) *only one negative eigenvalue, and the imaginary component also has only one positive eigenvalue.*

3. Comparing M. S. Livšic's theorem and Theorem IV.9.1 leads to the following result.

THEOREM 4.3 (M. G. KREĬN [9]). *In order that the system of root vectors of a dissipative operator A ($\in \mathfrak{S}_\infty$) with nuclear imaginary component be complete, it is necessary and sufficient that*

$$\int_0^\rho \frac{n(r;A)}{r}dr - \int_0^\rho \frac{n(r;A_{\mathscr{R}})}{r}\,dr = o(\rho), \tag{4.8}$$

as $\rho \to \infty$ avoiding some set of positive logarithmic length.

PROOF. If we denote by $N(\rho;A)$ and $N(\rho;A_{\mathscr{R}})$ respectively the two integrals in (4.8), then by Theorem IV.9.1, as $\rho \to \infty$ avoiding an appropriate set of finite logarithmic length,

$$N(\rho;A) - N(\rho;A_{\mathscr{R}}) = \frac{2}{\pi}\,a\rho + o(\rho),$$

where

$$a = \operatorname{sp} A_{\mathscr{J}} - \sum_{j=1}^{\infty} \operatorname{Im} \lambda_j(A).$$

Therefore the criterion (4.8) is equivalent to the criterion of M. S. Livšic (see Theorem 2.1). The theorem is proved.

We leave it to the reader to formulate the result which is obtained by comparing Theorems 4.1 and 4.3.

§5. Estimation of the growth of the resolvents of operators of various classes

1. If $A \in \mathfrak{S}_1$ then according to (IV.1.2) the following bound is valid for $D_A(\lambda) = \det(I - \lambda A)$:

$$|D_A(\lambda)| \leqq \prod_{j=1}^{\infty} (1 + |\lambda|\, s_j(A)).$$

This bound enables us to obtain an important bound for the operator $(I - \lambda A)^{-1}$.

THEOREM 5.1.[9] *If $A \in \mathfrak{S}_1$, then*

[9] The same result was obtained independently by V. B. Lidskiĭ [9]. His bound differed from (5.1) by an additional factor 2 in the right side. The authors obtained the bound (5.1) as the result of an analysis of a similar but less precise bound, obtained earlier by V. I. Macaev [4].

$$(5.1) \qquad |(I-\lambda A)^{-1}| \leqq \frac{1}{|D_A(\lambda)|}\prod_{j=1}^{\infty}(1+|\lambda|\, s_j(A)).$$

PROOF. Choosing arbitrary unit vectors ϕ, ψ and a positive ξ, we form the operator

$$A_1 = A + \xi(\cdot,\psi)\,\phi.$$

According to Corollary II.2.1, we have

$$s_{j+1}(A_1) \leqq s_j(A) \qquad (j=1,2,\cdots).$$

Since, moreover,

$$s_1(A_1) \leqq s_1(A) + \xi,$$

it follows that

$$(5.2) \qquad |D_{A_1}(\lambda)| \leqq (1+|\lambda|\,(s_1(A)+\xi))\prod_{j=1}^{\infty}(1+|\lambda|\, s_j(A)).$$

On the other hand,

$$1-\lambda\xi((I-\lambda A)^{-1}\phi,\psi) = \det((I-\lambda A_1)(I-\lambda A)^{-1}) = \frac{D_{A_1}(\lambda)}{D_A(\lambda)}.$$

Thus

$$|\lambda((I-\lambda A)^{-1}\phi,\psi)| \leqq \frac{1}{\xi} + \frac{1}{\xi}\frac{|D_{A_1}(\lambda)|}{|D_A(\lambda)|}.$$

We strengthen this inequality by replacing $|D_{A_1}(\lambda)|$ by the right side of (5.2). Then, letting ξ tend to infinity, we obtain

$$|((I-\lambda A)^{-1}\phi,\psi)| \leqq \frac{1}{|D_A(\lambda)|}\prod_{j=1}^{\infty}(1+|\lambda|\, s_j(A)).$$

Since the unit vectors ϕ and ψ are arbitrary, this inequality says the same thing as (5.1).

2. If A is a Volterra operator, we shall denote by $M_A(r)$ $(0 \leqq r < \infty)$ the monotonically increasing function[10] defined by

$$M_A(r) = \max_{|\lambda|=r}|(I-\lambda A)^{-1}| \left(= \max_{|\lambda|\leqq r}|(I-\lambda A)^{-1}|\right).$$

The growth of this function will be estimated later.

[10] The notation $M_A(r)$ is anologous to the current notation in the theory of functions.

If the Volterra operator A belongs to $\mathfrak{S}_1$, then $D_A(\lambda) = 1$, and the bound (5.1) yields

$$M_A(r) \leqq \prod_{j=1}^{\infty} (1 + rs_j(A)). \tag{5.3}$$

THEOREM 5.2 (V. I. MACAEV [4]).[11] *If A is a Volterra operator, and*

$$s_n(A) = O(n^{-1/p}) \qquad (= o(n^{-1/p})) \tag{5.4}$$

for some $p > 0$, then

$$\ln M_A(r) = O(r^{1/p}) \qquad (= o(r^{1/p})). \tag{5.5}$$

If $A \in \mathfrak{S}_p$, then also

$$\int_0^\infty \frac{\ln M_A(r)}{r^{p+1}} dr < \infty. \tag{5.6}$$

For the condition (5.4) replaced by the condition $A \in \mathfrak{S}_p$, the assertion (5.5), among others, was established by M. V. Keldyš (for a reference to this see Allahverdiev [1]). Other proofs of Keldyš' assertion appear in a paper by V. B. Lidskiĭ [9] and the book by Dunford and Schwartz [2].

PROOF. Let us first consider the case $p < 1$. In this case, on the basis of well-known theorems of the theory of entire functions, it follows from (5.4) that the entire function

$$F_A(z) = \prod_{j=1}^{\infty} (1 + zs_j(A))$$

has order of growth p and, depending upon whether the first or second of the assertions in (5.4) holds, will be of normal or of minimal type, i.e.,

$$\ln F_A(r) = O(r^{1/p}) \quad \text{or} \quad = o(r^{1/p}) \qquad (r \to \infty).$$

These relations, together with the bound (5.3), yield the relations (5.5). If $A \in \mathfrak{S}_p$, then according to (5.3) and Lemma IV.11.1 we shall have

$$\int_0^\infty \frac{\ln M_A(r)}{r^{p+1}} dr \leqq \int_0^\infty \frac{\ln F_A(r)}{r^{p+1}} dr = \beta_p |A|_p^p \qquad \left(\beta_p = \frac{\pi}{p \sin \pi p}\right).$$

Let us now consider the case $p \geqq 1$. We choose an integer q (> 0) such that $p_1 = p/q < 1$, and form the Volterra operator $B = A^q$. Since

[11] Macaev obtained this theorem in another way (as a consequence of certain deeper and more general results).

$$I - \lambda^q B = (I - \lambda A)(I + \lambda A + \cdots + \lambda^{q-1} A^{q-1}),$$
$$(I - \lambda A)^{-1} = (I + \lambda A + \cdots + \lambda^{q-1} A^{q-1})(I - \lambda^q B)^{-1},$$

it follows that

$$M_A(r) \leqq M_B(r^q) \sum_{k=0}^{q-1} r^k |A^k|. \tag{5.7}$$

On the other hand, by Corollary II.2.2 we have

$$s_n(B) = s_n(A^q) \leqq s^q_{[n/q]+1}(A) \qquad (n = 1, 2, \cdots),$$

and consequently, if one of the relations (5.4) holds, then

$$s_n(B) = O(n^{-1/p_1}) \qquad (\text{or} = o(n^{-1/p_1})).$$

But then, by what was already proved,

$$\ln M_B(r) = O(r^{p/q}) \qquad (= o(r^{p/q})) \qquad (r \to \infty).$$

In conjunction with (5.7), this again yields (5.4).

If $A \in \mathfrak{S}_p$, then $B \in \mathfrak{S}_{p_1}$, and by what was already proved we will have

$$\int_0^\infty \frac{\ln M_B(r^q)}{r^{p+1}} dr = \frac{1}{q} \int_0^\infty \frac{\ln M_B(r)}{r^{p_1+1}} dr < \infty.$$

Taking (5.7) into account, we obtain (5.6). The theorem is proved.

Of course, the first part of Theorem 5.2 can without difficulty be generalized to the case where, instead of the relations (5.4), one considers the relations

$$s_n(A) = O(n^{-1/p} L(n) \qquad (\text{or} = o(n^{-1/p} L(n))),$$

where $L(\nu)$ $(1 \leqq \nu < \infty)$ is a slowly varying function.

We also note that the bounds (5.3) and (5.5) enable us to obtain a bound for the Fredholm resolvent $A(\lambda) = A(I - \lambda A)^{-1}$ of a Volterra operator A in the norm corresponding to an s.n. ideal $\mathfrak{S}$, since

$$|A(\lambda)|_{\mathfrak{S}} \leqq |A|_{\mathfrak{S}} |(I - \lambda A)^{-1}|.$$

§6. Theorems on the completeness of the systems of root vectors of higher classes of operators

1. By a well-known theorem of Hausdorff (see Stone [1]), for every operator $A \in \mathfrak{R}$ the set of values of its quadratic form (Af, f) on the unit sphere $|f| = 1$ is convex.

We shall denote the closure of this set by W_A. With the help of

Theorem IV.4.1, one can prove the following general assertion (see Stone [1]).

Lemma 6.1. *Let $A \in \mathfrak{R}$ and $\lambda \notin W_A$; then λ is a regular point of the operator A and*

$$|(A - \lambda I)^{-1}| \leqq \frac{1}{d(\lambda, W_A)},$$

where $d(\lambda, W_A)$ denotes the distance from the point λ to the set W_A.

Proof. Let a be the point of W_A closest to λ. Then λ and W_A will lie on opposite sides of the line passing through the point a and perpendicular to the line segment joining the points λ and a. Then the region W_B corresponding to the operator $B = (A - aI)\, e^{i\alpha}$, where $\alpha = -\arg(\lambda - a) - \pi/2$, is obtained from the region W_A by means of the parallel displacement $z \to z - a$ and the rotation $z \to ze^{i\alpha}$, and will thus lie in the upper halfplane and meet the real axis at the point O. Under the indicated motions, the point λ goes over into the point $\mu = (\lambda - a)\, e^{i\alpha} = -|\lambda - a|\, i$. Since B is a dissipative operator, we have

$$|(A - \lambda I)^{-1}| = |(B - \mu I)^{-1}| \leqq \frac{1}{|\operatorname{Im} \mu|} = \frac{1}{|\lambda - a|}.$$

The lemma is proved.

2. If $A \in \mathfrak{R}$, then obviously the closure $\widetilde{W}_A$ of the set of all the values (Af, f) $(f \in \mathfrak{H})$ consists of the points z of the form ζt, where $\zeta \in W_A$ and $0 \leqq t < \infty$. From the convexity of W_A it follows that either $\widetilde{W}_A$ coincides with some sector with apex at the origin of coordinates and angular measure $\theta_A \leqq \pi$, or it coincides with the entire plane (and in this case we shall write $\theta_A = 2\pi$).

Theorem 6.1. *Suppose that the following two conditions hold for $A \in \mathfrak{S}_\infty$:*

$$1) \qquad \theta_A \leqq \pi/p, \quad \text{where} \quad p \geqq 1;$$

$$(6.1) \qquad 2) \qquad s_n(A) = o(n^{-1/p}) \qquad (n \to \infty).$$

Then the system of all root vectors of the operator A is complete in $\mathfrak{H}$.

Obviously condition 2) of the theorem will be fulfilled if $A \in \mathfrak{S}_p$ $(1 \leqq p < \infty)$. With this restriction, the theorem is stated in the report by M. V. Keldyš and V. B. Lidskiĭ [1], and is a natural generalization of theorems established earlier by Lidskiĭ for the cases $p = 1$ and $p = 2$. The bound (5.5) has made it possible to strengthen this theorem somewhat.

Proof. Without loss of generality we may assume that the operator A is dissipative. The properties 1) and 2) of an operator A, formulated in the statement of the theorem, are obviously invariant with respect to the operation of orthogonal projection ($A \to PAP$). Therefore, on the basis of Lemma 1.2, to prove the theorem it suffices to show that a dissipative Volterra operator A which satisfies conditions 1) and 2) is equal to zero.

By Theorem 5.2 we have, for a Volterra operator A,

$$\ln|A(z)| = \ln|A(I - zA)^{-1}| = o(|z|^p) \qquad (z \to \infty). \tag{6.2}$$

On the other hand, since $W_A \subset \widetilde{W}_A$, according to Lemma 6.1

$$|(A - \lambda I)^{-1}| \leqq 1/d(\lambda, \widetilde{W}_A),$$

and consequently

$$|(I - zA)^{-1}| \leqq 1/d(1/z, \widetilde{W}_A)|z|. \tag{6.3}$$

Let C_1 and C_2 be discs of radius 1 which are tangent to the first and second sides, respectively, of the sector $\widetilde{W}_A$ at the point $\lambda = 0$ and which lie outside this sector. We denote the union of C_1 and C_2 by C. It is easily seen that if $\lambda \in C$, then, given $|\lambda|$, the value $d(\lambda, \widetilde{W}_A)$ will be smallest when λ lies on the boundary of C and when the chord, directed from the point O to λ, forms an acute angle with one of the sides of the sector $\widetilde{W}_A$. For this point λ it is easily seen that

$$d(\lambda, \widetilde{W}_A) = |\lambda|^2/2.$$

It follows that

$$d(\lambda, \widetilde{W}_A) \geqq |\lambda|^2/2 \qquad (\lambda \in C). \tag{6.4}$$

The transformation $z = 1/\lambda$ maps the region C into the exterior of some sector W' of angular measure θ_A. Consequently, according to (6.3) and (6.4),

$$|(I - zA)^{-1}| \leqq 2|z| \qquad (z \notin W')$$

and

$$\sup_{z \notin W'} |A(z)| = \sup_{z \notin W'} \left| \frac{(I - zA)^{-1} - I}{z} \right| < \infty.$$

In particular, the entire operator-function $A(z)$ is bounded on the sides of the sector W'. Since, moreover, (6.2) is fulfilled, by the Phragmén-Lindelöf Theorem (see Titchmarsh [1]) the function $A(z)$

is also bounded inside the sector W' and consequently over the entire plane. But then, by Liouville's theorem,

$$A(z) = A + zA^2 + \cdots = A.$$

Hence $A^2 = 0$, i.e., if $g = Af$, then $Ag = 0$. But since the operator A is dissipative, also $A^*g = 0$, i.e. $A^*Af = 0$ and $Af = 0$ $(f \in \mathfrak{H})$. The theorem is proved.

REMARK 6.1. We shall show that if Theorem 6.1 holds for operators A with $\theta_A \leqq \theta_0$ $(0 < \theta_0 < \pi)$, then its validity for operators A with $\theta_A > \theta_0$ follows. In fact, let A be an operator which satisfies the condition

$$|\arg(Af,f)| \leqq \theta/2 \qquad (f \in \mathfrak{H}). \tag{6.5}$$

In this case it is natural to define the power A^κ $(0 < \kappa < 1)$ by (see Kato [1], Phillips [1], Macaev and Palant [1])

$$A^\kappa = \frac{\sin \pi\kappa}{\pi} \int_0^\infty \lambda^\kappa \left((A + \lambda I)^{-1} - \frac{1}{\lambda} I \right) d\lambda.$$

It is fairly simple to verify that

$$|\arg(A^\kappa f,f)| \leqq \theta\kappa/2.$$

One verifies directly that if $\lambda_0 = |\lambda_0| e^{i\phi}$ $(\phi < \theta/2)$ is a normal eigenvalue of the operator A then $\lambda_0^\kappa = |\lambda_0|^\kappa e^{i\kappa\phi}$ is a normal eigenvalue of the operator A^κ, and the corresponding root subspaces $\mathfrak{L}_{\lambda_0}(a)$ and $\mathfrak{L}_{\lambda_0^\kappa}(A^\kappa)$ coincide.

One can also assert (see Macaev and Palant [1]) that if $A \in \mathfrak{S}_\infty$, then $A^\kappa \in \mathfrak{S}_\infty$ and

$$s_{2n}(A^\kappa) \leqq C s_n^\kappa(A) \qquad (n = 1, 2, \cdots),$$

where C is a constant depending only upon κ. Therefore if the condition (6.5) is fulfilled for the operator A with $\theta = \theta_A$ and if $s_n(A) = o(n^{-1/p})$ $(n \to \infty)$ for $p = \pi/2\theta$, then for the operator $B = A^\kappa$ we will have $\theta_B \leqq \kappa\theta_A$ and $s_n(B) = o(n^{-\kappa/p})$ $(n \to \infty)$. Consequently every such operator will satisfy the hypotheses of Theorem 6.1, which explains the assertion made.

REMARK 6.2. Due to lack of space, we have not discussed the important investigations of V. B. Lidskiĭ [7-9] on the possibility of summing, by Abel's generalized method, formal expansions of an arbitrary element $f \in \mathfrak{H}$ in the root vectors of an operator A satisfying conditions close to those of Theorem 6.1. In this connection, see also the paper by Macaev and Markus [5].

3. Theorem IV.11.2 of V. I. Macaev and the authors enables us to obtain the following improvement of Theorem 6.1 for the case $p > 1$.

THEOREM 6.2. *Let $A \in \mathfrak{S}_\infty$ and suppose that the following two conditions are fulfilled:*

1) $\theta_A \leqq \pi/p$, *where* $p > 1$;

2) *for some* α *one has, for the operator* $B = [e^{i\alpha}A]_{\mathscr{I}}$,

$$s_n(B) = o(n^{-1/p}) \qquad (n \to \infty). \tag{6.6}$$

Then the system of all root vectors of the operator A is complete in $\mathfrak{H}$.

The proof of this theorem repeats word for word the proof of Theorem 6.1, if one notes additionally that for a Volterra operator A the relation (6.6) implies the relation (6.1).

Let us make some clarification of Theorem 6.2. Suppose, for definiteness, that the operator A satisfies the condition (6.5). It is easily seen that this condition is equivalent to the following:

$$|(A_{\mathscr{I}}f,f)| \leqq (\tan\theta/2)(A_{\mathscr{R}}f,f),$$

and so

$$s_n(A_{\mathscr{I}}) \leqq (\tan\theta/2)\, s_n(A_{\mathscr{R}}) \qquad (n = 1, 2, \cdots). \tag{6.7}$$

Therefore, if the hypotheses of Theorem 6.2 are fulfilled for the given operator A with $\alpha = \pi/2$ (in this case the condition (6.6) assumes the form $s_n(A_{\mathscr{R}}) = o(n^{-1/p})$ $(n \to \infty)$), then by virtue of (6.7) the hypothesis of Theorem 6.1 will also be fulfilled. Thus in this case Theorem 6.2 will not give anything new in comparison with Theorem 6.1. Conversely, for $\alpha = 0$ the condition (6.6) says that $s_n(A_{\mathscr{I}}) = o(n^{-1/p})$, and in this case we obtain an explicit strengthening of Theorem 6.1. It is easily seen that in general one obtains a strengthening of Theorem 6.1 for values of α from the interval $|\alpha| \leqq \theta_A$.

Let us further make it clear that one has to exclude the case $p = 1$ in Theorem 6.2, since for a Volterra operator A the relation

$$s_n(A_{\mathscr{I}}) = o(n^{-1}) \qquad (n \to \infty)$$

does not imply the analogous relation for the operator itself (see the example in §7.4, Chapter IV).

Theorem 6.1 and a fortiori Theorem 6.2 is "precise": the condition (6.6) in it cannot be replaced by the condition

$$s_n(A) = O(n^{-1/p}) \qquad (n \to \infty).$$

Indeed, let us consider the Volterra operator $T = J^{\nu}$ in $L_2(0,1)$:

$$(J^{\nu}f)(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s)\,ds \qquad (f \in L_2(0,1)),$$

where $\nu = 1/p$. Then one can show (see Remark 6.1) that on the one hand

$$|\arg(Tf,f)| \leqq \pi\nu/2,$$

and on the other hand

$$s_n(T) = O(n^{-\nu}) \qquad (n \to \infty);$$

consequently

$$s_n(\cos\alpha\, T_{\mathscr{R}} + \sin\alpha\, T_{\mathscr{J}}) = O(n^{-\nu}) \qquad (n \to \infty).$$

The authors owe this example to B. Ja. Levin and V. I. Macaev.

As a simple corollary of Theorem 6.2, we obtain

THEOREM 6.3. *Let $A \in \mathfrak{S}_\infty$ be an operator with nonnegative Hermitian components and suppose that for some α the operator $B = [e^{i\alpha}A]_{\mathscr{J}}$ belongs to $\mathfrak{S}_2$ or (a more general condition)*

$$s_n(B) = o(1/\sqrt{n}) \qquad (n \to \infty).$$

Then the system of root vectors of A is complete in $\mathfrak{H}$.

Indeed, if the components $A_{\mathscr{R}}$ and $A_{\mathscr{J}}$ are nonnegative, then all the values of the quadratic form

$$(Af,f) = (A_{\mathscr{R}}f,f) + (A_{\mathscr{J}}f,f)$$

belong to the first quadrant, and consequently $\theta_A \leqq \pi/2$.

The last theorem is antedated by a simpler theorem of V. B. Lidskiĭ [4].

Suppose that $A \in \mathfrak{S}_2$ and that its Hermitian components are nonnegative; then the system of root vectors of the operator A is complete in $\mathfrak{H}$.

We present an elementary proof of this theorem, found by V. B. Lidskiĭ. By virtue of Lemma 1.2, the proof of the theorem reduces to proving that there does not exist a simple Volterra operator which satisfies the conditions of the theorem.

Let us assume the contrary, namely that some operator A which satisfies the conditions of the theorem is simple and Volterra. Since by hypothesis $A \in \mathfrak{S}_2$, we have $A^2 \in \mathfrak{S}_1$. On the other hand, A^2, along with A, is a Volterra operator, so that

$$\operatorname{sp} A^2 = \operatorname{sp}(A_{\mathscr{R}}^2 - A_{\mathscr{J}}^2) + 2i\operatorname{sp}(A_{\mathscr{R}}A_{\mathscr{J}}) = 0.$$

Since, according to Theorem III.8.3, the trace $\mathrm{sp}(A_{\mathscr{R}}A_{\mathscr{I}})$ is a real number, we conclude that $\mathrm{sp}(A_{\mathscr{R}}A_{\mathscr{I}}) = 0$. But then, by the same theorem, the operators $A_{\mathscr{R}}$ and $A_{\mathscr{I}}$ commute (and moreover $A_{\mathscr{R}}A_{\mathscr{I}} = 0$). Thus the operator A is normal and consequently cannot be a Volterra operator (A equal to zero is excluded by the condition that A be simple). The contradiction which has been obtained proves Lidskiĭ's theorem. Following the appearance of L. A. Sahnovič's theorem (see §11.3, Chapter IV), both of these authors noted that the condition $A \in \mathfrak{S}_2$ can be replaced by the condition $[e^{i\alpha}A]_{\mathscr{I}} \in \mathfrak{S}_2$.

4. Drawing upon rather complicated function-theoretic methods, V. I. Macaev established the following result.

THEOREM 6.4 (V. I. MACAEV [5]). *The system of root vectors of a completely continuous dissipative operator A is complete in $\mathfrak{H}$ whenever*

$$\lim_{\rho\to\infty} \frac{n(\rho; A_{\mathscr{I}})}{\rho^2} = 0 \tag{6.8}$$

and at least one of the following two conditions is fulfilled:

$$\lim_{\rho\to\infty} \frac{n_+(\rho; A_{\mathscr{R}})}{\rho} = 0 \quad \textit{or} \quad \lim_{\rho\to\infty} \frac{n_-(\rho; A_{\mathscr{R}})}{\rho} = 0. \tag{6.9}$$

Theorem IV.11.2 allows the condition (6.8) to be replaced by a more general one, namely the condition

$$\lim_{\rho\to\infty} \frac{n(\rho; \cos\alpha A_{\mathscr{R}} + \sin\alpha A_{\mathscr{I}})}{\rho} = 0 \tag{6.10}$$

for some α.

Formulated in this way, the theorem represents a substantial sharpening of Theorem 6.3.

In particular, if we take $\alpha = 0$ in (6.10), then we arrive at this test:

THEOREM 6.5 (MACAEV [5]). *The system of root vectors of a completely continuous dissipative operator A is complete in $\mathfrak{H}$ whenever at least one of the following two pairs of conditions is fulfilled:*

$$\lim_{\rho\to\infty} \frac{n_+(\rho; A_{\mathscr{R}})}{\rho} = 0 \quad \textit{and} \quad \lim_{\rho\to\infty} \frac{n_-(\rho; A_{\mathscr{R}})}{\rho^2} = 0$$

or

$$\lim_{\rho\to\infty} \frac{n_+(\rho; A_{\mathscr{R}})}{\rho^2} = 0 \quad \textit{and} \quad \lim_{\rho\to\infty} \frac{n_-(\rho; A_{\mathscr{R}})}{\rho} = 0.$$

§7. Two lemmas on the resolvents of normal operators

The first of these lemmas was, apparently, already used by M. V. Keldyš [1] (cf. also D. È. Allahverdiev [1]). The second lemma, as formulated here, is presented for the first time, although its proof is entirely taken over from a paper by the authors [1]. There is a close connection between these two lemmas.

LEMMA 7.1. *Let $H \in \mathfrak{S}_\infty$ be a complete*[12] *normal operator, all of whose characteristic numbers, except possibly a finite number of them, lie outside the sector*

$$\theta_1 < \arg \zeta < \theta_2 \qquad (0 \leqq \theta_1 < \theta_2 \leqq 2\pi),$$

and let $T \in \mathfrak{S}_\infty$. Then for any $\epsilon > 0$ the limit relation

$$\lim_{\zeta \to \infty} | T(I - \zeta H)^{-1} | = 0$$

holds uniformly in the sector F_ϵ:

$$\theta_1 + \epsilon \leqq \arg \zeta \leqq \theta_2 - \epsilon.$$

PROOF. As is well known,[13] for a normal operator H one has

$$| (H - \lambda I)^{-1} | = 1/d(\lambda),$$

where $d(\lambda)$ is the distance from the number λ to the spectrum of H. Consequently

$$| (I - \zeta H)^{-1} | = | \zeta^{-1} | / d(1/\zeta).$$

For sufficiently large $\zeta \in F_\epsilon$ the quantity $d(1/\zeta)$ will be smallest, for fixed modulus $|\zeta|$, when the point ζ lies on the sides of the sector F_ϵ. In this case it is easily seen that

$$| \zeta^{-1} | / d(1/\zeta) \leqq 1/\sin \epsilon.$$

Thus for sufficiently large $\zeta \in F_\epsilon$

$$(7.1) \qquad | (I - \zeta H)^{-1} | \leqq 1/\sin \epsilon.$$

Choosing an arbitrary $\delta > 0$, we represent the operator T in the form $T = K + M$, where K is a finite-dimensional operator,

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[12] A normal operator H is said to be *complete* if it vanishes only on the zero vector.

[13] This is easily deduced from the spectral representation of a normal operator.

$$K = \sum_{j=1}^{n} (\cdot, \psi_j)\, \phi_j \qquad (|\phi_j| = 1, j = 1, 2, \cdots n),$$

and the operator M has small norm, namely

$$|M| < \frac{\delta}{2} \sin \epsilon.$$

According to the bound (7.1) we have, for sufficiently large $\zeta \in F_\epsilon$,

$$|M(I - \zeta H)^{-1}| < |M|/\sin \epsilon < \delta/2.$$

Since by hypothesis H is a complete normal operator from $\mathfrak{S}_\infty$, one has

$$(I - \zeta H)^{-1} = \sum_{j=1}^{\infty} \frac{\mu_j(\cdot, e_j)\, e_j}{\mu_j - \zeta},$$

where $\{e_j\}_1^\infty$ is a complete orthonormal sequence of eigenvectors of H, and $\{\mu_j\}_1^\infty$ is the corresponding sequence of characteristic numbers of H.

We choose a number N, and then a number $R > 0$ so large that

$$\left(\sum_{j=N+1}^{\infty} |(\psi_k, e_j)|^2 \right)^{1/2} < \frac{\delta \sin \epsilon}{4n} \qquad (k = 1, 2, \cdots, n), \tag{7.2}$$

$$\left(\sum_{j=1}^{N} \left| \frac{\mu_j}{\mu_j - \zeta} \right|^2 |(\psi_k, e_j)|^2 \right)^{1/2} < \frac{\delta}{4n} \qquad (|\zeta| \geqq R; k = 1, 2, \cdots, n). \tag{7.3}$$

Taking into account that for any $f \in \mathfrak{H}$

$$K(I - \zeta H)^{-1} f = \sum_{j=1}^{\infty} \sum_{k=1}^{n} \frac{\mu_j(\psi_k, e_j)(f, e_j)}{\mu_j - \zeta}\, \phi_k,$$

we obtain

$$|K(I - \zeta H)^{-1} f| \\ \leqq \sum_{k=1}^{n} \left(\sum_{j=1}^{\infty} \left| \frac{\mu_j}{\mu_j - \zeta} (\psi_k, e_j) \right|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |(f, e_j)|^2 \right)^{1/2},$$

and so

$$|K(I - \zeta H)^{-1} f| \leqq |f| \sum_{k=1}^{n} \left(\sum_{j=1}^{\infty} \frac{|\mu_j|^2}{|\mu_j - \zeta|^2} |(\psi_k, e_j)|^2 \right)^{1/2}.$$

Since

$$\frac{|\mu_j|}{|\mu_j - \zeta|} = \frac{|\zeta^{-1}|}{|\zeta^{-1} - \mu_j^{-1}|} \leqq \frac{|\zeta^{-1}|}{d(1/\zeta)} \leqq \frac{1}{\sin \epsilon} \qquad (\zeta \in F_\epsilon; |\zeta| \geqq R),$$

we conclude on the basis of (7.2) and (7.3) that

$$|K(I-\zeta H)^{-1}f| < \delta|f|/2 \qquad (\zeta \in F_\epsilon; |\zeta| \geqq R).$$

Thus

$$|T(I-\zeta H)^{-1}| \leqq |K(I-\zeta H)^{-1}| + |M(I-\zeta H)^{-1}| < \epsilon$$
$$(\zeta \in F_\epsilon; |\zeta| \geqq R).$$

The lemma is proved.

Lemma 7.1 remains valid in the case where H is an arbitrary (even unbounded) normal operator which does not vanish on any nonnull vector. In this general case the lemma, in turn, follows from a more precise result. We shall formulate the latter only for the case of a selfadjoint operator H.

LEMMA 7.2. *Let H be an arbitrary selfadjoint operator, $\mathfrak{S}$ any separable s.n. ideal, and B a closed operator with the following two properties*:

a) *the domain $\mathfrak{D}(B)$ of the operator B contains the domain $\mathfrak{D}(H)$ of the operator H*;

b) *$B(H-\lambda_0 I)^{-1} \in \mathfrak{S}$ for at least one regular point λ_0 of the operator H.*

Then for any sector F with apex at the origin of coordinates which does not contain any point λ ($\lambda \neq 0$) of the spectrum of the operator H, the limit relation

$$\lim_{\lambda \in F;\, \lambda \to \infty} |B(H-\lambda I)^{-1}|_{\mathfrak{S}} = 0 \tag{7.4}$$

holds uniformly.

Let us clarify the connection between Lemmas 7.1 and 7.2. We remark that if H_1 is any selfadjoint operator which vanishes only at zero, and if $T \in \mathfrak{S}_\infty$, then the operator $T(I-\zeta H_1)^{-1}$ can be represented in the form

$$T(I-\zeta H_1)^{-1} = B(H-\zeta I)^{-1},$$

where $B = TH_1^{-1}$ and $H = H_1^{-1}$. It is easily seen that the operators H and B satisfy the conditions of Lemma 7.2 for $\mathfrak{S} = \mathfrak{S}_\infty$.

PROOF. From the equality

$$BR(\lambda) = BR(\lambda_0) + (\lambda - \lambda_0)\, BR(\lambda_0)\, R(\lambda),$$

where $R(\lambda) = (H-\lambda I)^{-1}$, it follows that if the operator B has the property b), then $BR(\lambda) \in \mathfrak{S}$ for any regular point λ of the operator H.

Let us assume, for simplicity, that the sector F has the form

$$\epsilon \leqq \arg \lambda \leqq \pi - \epsilon \qquad (\epsilon > 0).$$

The operator $BR(z)$ can be represented in the form

(7.5) $$BR(z) = BR(i) + (z - i)\, BR(i)\, R(z) = BR(i)\, Q(z),$$

where $Q(z) = I + (z - i)\, R(z)$. The spectral representation of the operator $Q(z)$ gives the formula

$$Q(z) = \int_{-\infty}^{\infty} \left(1 + \frac{z-i}{\lambda - z}\right) dE_\lambda = \int_{-\infty}^{\infty} \frac{\lambda - i}{\lambda - z} dE_\lambda,$$

where E_λ is the spectral function of the operator H. It is easily seen that if λ ranges over the spectrum $\sigma(H)$ of H and z ranges over that part of the vector F lying in the region $|z| > 1$, then the function $|(\lambda - i)/(\lambda - z)|$ is bounded by some constant γ, and consequently

$$|Q(z)| \leqq \gamma \qquad (z \in F; |z| > 1).$$

We now observe that for any $f \in \mathfrak{H}$

(7.6) $$\lim_{z \in F;\, z \to \infty} |Q^*(z) f| = 0.$$

In fact,

$$\begin{aligned} |Q(z) f|^2 &= \int_{-\infty}^{\infty} \left|\frac{\lambda + i}{\lambda - z}\right|^2 d(E_\lambda f, f) \\ &\leqq \int_{-N}^{N} \left|\frac{\lambda + i}{\lambda - z}\right|^2 d(E_\lambda f, f) + \gamma^2 \left(\int_{-\infty}^{-N} + \int_{N}^{\infty}\right) d(E_\lambda f, f). \end{aligned}$$

For a given f, the second term can be made arbitrarily small by a suitably choice of N, and for fixed N the first term tends to zero as $z \to \infty$.

For any finite-dimensional operator

$$K = \sum_{j=1}^{n} (\cdot, \psi_j)\, \phi_j$$

we have

$$KQ(z) = \sum_{j=1}^{n} (\cdot, Q^*(z)\, \psi_j)\, \phi_j,$$

and consequently

$$|KQ(z)|_{\mathfrak{S}} \leq \sum_{j=1}^{n} |Q^*(z)\psi_j|\,|\phi_j|.$$

It follows from (7.6) that

$$\lim_{z\in F;\, z\to\infty} |KQ(z)|_{\mathfrak{S}} = 0.$$

Finally, bearing in mind the equality (7.5) and the fact that the operator $BR(i)$ $(\in \mathfrak{S})$ can be approximated arbitrarily closely in $\mathfrak{S}$-norm by finite-dimensional operators, we obtain the relation (7.4). The lemma is proved.

§8. Theorems on the completeness of the system of root vectors of a weakly perturbed selfadjoint operator

1. In §§2-6 we introduced various tests for the completeness of the system of root vectors of a nonselfadjoint operator A. All of these included the requirement that A be dissipative or the more restrictive requirement $\theta_A < \pi$.

At the same time, already in the first fundamental work of M. V. Keldyš [1] on the theory of the eigenvectors and associated vectors of nonselfadjoint operator bundles, an important test for the completeness of the system of root vectors of a nonselfadjoint operator A was obtained, in which the above-mentioned requirement played no role whatsoever.

Following M. V. Keldyš, we shall say that an operator $A \in \mathfrak{S}_\infty$ has *finite order*, if it belongs to some space $\mathfrak{S}_p$ $(p < \infty)$, i.e. if

$$\sum_{n=1}^{\infty} s_n^p(A) < \infty$$

for some p $(< \infty)$. We shall call the infimum of those numbers p for which this relation holds the *order* of the operator A, and denote it by $p(A)$.

Since, by definition, $p(A)$ coincides with the order of the sequence $\{s_n^{-1}(A)\}_{n=0}^{\infty}$, we have (see Levin [1], Chapter I §4)

$$p(A) = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln(1/s_n(A))}.$$

Obviously

$$p(A) = p(A^*) \quad \text{and} \quad p(A+B) \leq \max\{p(A), p(B)\};$$

in particular, $p(A) = \max\{p(A_{\mathcal{J}}), p(A_{\mathcal{R}})\}$.

Some of the results of M. V. Keldyš presented below could be strengthened by using the following result.

LEMMA 8.1 (GOHBERG AND KREĬN [4]). *If one of the Hermitian components of a Volterra operator A has finite order $\geqq 1$, then the operator A has the same order.*

This result was first proved by the authors [4] (by means of the theory of the abstract triangular integral). Of course, it is a corollary of the stronger Theorem IV.11.1.

THEOREM 8.1 (KELDYŠ [1]). *Let*

$$A = H(I+S),$$

where $H = H^*$ and $p(H) < \infty$, and $S \in \mathfrak{S}_\infty$. If the operator A vanishes only at zero, then the system of its root vectors is complete in $\mathfrak{H}$. For any arbitrarily small ϵ (>0), all eigenvalues of A, except possibly a finite number of them, lie in the sectors

$$(8.1) \qquad -\epsilon < \arg\lambda < \epsilon; \quad \pi - \epsilon < \arg\lambda < \pi + \epsilon$$

of angular measure 2ϵ. If the operator H has only a finite number of negative (positive) eigenvalues, then A has no more that a finite number of eigenvalues in the sector

$$\pi - \epsilon < \arg\lambda < \pi + \epsilon \qquad (-\epsilon < \arg\lambda < \epsilon).$$

PROOF.[14] It follows from the conditions of the theorem that the operator $I + S$ vanishes only at zero, and consequently is invertible since S is completely continuous. It follows in turn that the operator H also vanishes only at zero, and consequently is complete.

According to Lemma 7.1, for any $\epsilon > 0$ there exists a number $r > 0$ such that in the region F_ϵ,

$$(8.2) \qquad \epsilon \leqq |\arg\zeta| \leqq \pi - \epsilon, \qquad |\zeta| \geqq r$$

we will have

$$|T(I-\zeta H)^{-1}| < q < 1,$$

where $T = I - (I+S)^{-1} \in \mathfrak{S}_\infty$.

---

[14] This proof differs inessentially from the proof by Keldyš (see Keldyš and Lidskiĭ [1]).

If the operator H has only a finite number of negative eigenvalues, then according to the same Lemma 7.1 the region F_ϵ can be enlarged, namely its definition can be replaced by

$$\epsilon \leqq \arg \zeta \leqq 2\pi - \epsilon, \qquad |\zeta| \geqq r. \tag{8.3}$$

In what follows we shall assume that in the general case the region F_ϵ is defined by (8.2), and in the special case for which H has only a finite number of negative eigenvalues, by (8.3).

Since the operator $I - \zeta A$ is representable in the form

$$I - \zeta A = [(I+S)^{-1} - \zeta H](I+S) = (I - T - \zeta H)(I+S),$$

hence in the form

$$I - \zeta A = (I - T(I - \zeta H)^{-1})(I - \zeta H)(I+S),$$

it is invertible in the region F_ϵ. Namely,

$$(I - \zeta A)^{-1} = (I+S)^{-1}(I - \zeta H)^{-1}\sum_{n=0}^{\infty}(T(I - \zeta H)^{-1})^n. \tag{8.4}$$

Thus we have established the invertibility of the operator $I - \zeta A$ in the region $\zeta \in F_\epsilon$, and hence proved the last two assertions of the theorem.[15]

At the same time, from the equality (8.4) and the bound

$$|(I - \zeta H)^{-1}| \leqq \frac{1}{\sin \epsilon} \qquad (\zeta \in F_\epsilon)$$

we obtain the bound

$$|(I - \zeta A)^{-1}| \leqq \frac{|(I+S)^{-1}|}{(1-q)\sin\epsilon} \,(< \infty) \qquad (\zeta \in F_\epsilon). \tag{8.5}$$

We now proceed to the proof of the first assertion of the theorem.

As usual, we denote by $\mathfrak{E}$ the closed linear hull of all the root vectors of the operator A. We have to prove that $\mathfrak{E} = \mathfrak{H}$. Let us assume the contrary, that $\mathfrak{E} \neq \mathfrak{H}$.

We denote by P the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{E}$. According to Lemma I.4.2, the operator $A_1 = QAQ$ $(Q = I - P)$ is a

[15] We remark that the preceding argument, and consequently these two assertions, remain valid for any bounded (and even unbounded) complete selfadjoint operator H.

Volterra operator, and $(I-\zeta A_1)^{-1}$ is an entire operator-function.

According to result 2 of §1, Chapter I,

$$(I-\zeta A_1)^{-1}=Q(I-\zeta A)^{-1}Q+P \qquad (\zeta\in F_\epsilon),$$

and consequently by virtue of (8.5) the operator-function $(I-\zeta A_1)^{-1}$ is bounded in the region F_ϵ. On the other hand, $A_1\in\mathfrak{S}_p$ for $p>p(H)$, and by Theorem 5.2

$$\ln|(I-\zeta A_1)^{-1}|=o(|\zeta|^p) \qquad (\zeta\to\infty). \tag{8.6}$$

We now choose $\epsilon<\pi/p$. Since the operator-function $(I-\zeta A_1)^{-1}$ is bounded on the sides of the sectors

$$|\arg\zeta|\leqq\epsilon, \qquad |\pi-\arg\zeta|\leqq\epsilon,$$

by the Phragmén-Lindelöf theorem it follows from (8.6) that the operator-function $(I-\zeta A_1)^{-1}$ is bounded inside these sectors. Thus the entire operator-function $(I-\zeta A_1)^{-1}$ is bounded over the entire complex plane. It follows that $(I-\zeta A_1)^{-1}\equiv I$ and

$$A_1=QAQ=QA=0.$$

But then $A^*Q=0$, i.e. the operator A^* annihilates the entire subspace $Q\mathfrak{H}$. This leads to a contradiction, since the operator $A^*=(I+S^*)H$ vanishes only at zero. The theorem is proved.

REMARK 8.1. When the conditions of Theorem 8.1 are fulfilled, one can also assert that *the system of root vectors of the adjoint A^* is complete in $\mathfrak{H}$.*[16] In fact, the operator $A_1=H(I+S^*)$ satisfies all the conditions of Theorem 8.1, and the operator $A^*=(I+S^*)H$ is similar to it: $A^*=(I+S^*)A_1(I+S^*)^{-1}$.

We note, more generally, that if an operator A has a complete system of root vectors, then every operator similar to A, i.e. every operator $W^{-1}AW$ $(W,W^{-1}\in\mathfrak{R})$, has the same property. It follows that Theorem 8.1 can be generalized to the case of an operator A of the form $(I+S_1)H(I+S_2)$, where $S_j\in\mathfrak{S}_\infty$ $(j=1,2)$ and the operators $I+S_j$ $(j=1,2)$ are invertible.

---

[16] It does not generally follow from the completeness of the system of root vectors of an operator A that the system of root vectors of the operator A^* is complete. As was observed by A. S. Markus, from results of H. Hamburger [1] there follows the existence of an operator $A\in\mathfrak{S}_1$ which has a complete system of eigenvectors, corresponding to the nonzero eigenvalues, while at the same time the orthogonal complement in $\mathfrak{R}(A^*)$ of the closed linear hull of all the root vectors of the operator A^* is infinite-dimensional.

In fact, if we put $I+S=(I+S_2)(I+S_1)$ and $A_1=H(I+S)$, then $A=W^{-1}A_1W$, where $W=(I+S_1)^{-1}$.

2. Theorem 8.1 is a special corollary of a general theorem of M. V. Keldyš on the n-fold completeness of the system of eigenvectors and associated vectors of an operator bundle of a specified structure. The following §9 is entirely devoted to this general theorem. For the special case $n=1$ which interests us here, the theorem can be formulated as follows.

THEOREM 8.2 (KELDYŠ [1]). *The system of eigenvectors and associated vectors of the linear bundle*

$$L(\lambda)=I-T-\lambda H \tag{8.7}$$

is complete in $\mathfrak{H}$, whenever H is a complete selfadjoint operator of finite order, and T is an arbitrary operator from $\mathfrak{S}_\infty$.

Let us clarify the new concepts which appear in the statement of this theorem.

A number λ_0 is called a *characteristic number* of the bundle $L(\lambda)$ if the equation $L(\lambda_0)\phi=0$ has a nonnull solution. Such a solution is called an *eigenvector* of the bundle $L(\lambda)$, corresponding to the characteristic number λ_0.

The vectors $\phi_1,\phi_2,\cdots,\phi_k$ are said to be *associated* with the eigenvector ϕ_0 $(L(\lambda_0)\phi_0=0)$, if

$$\phi_r-T\phi_r-\lambda_0H\phi_r=H\phi_{r-1} \qquad (r=1,2,\cdots,k). \tag{8.8}$$

We shall now show that Theorems 8.1 and 8.2 are equivalent, i.e., each is a consequence of the other. To do this we shall first clarify a number of general properties of the bundle (8.7) under the sole assumption that H is a complete selfadjoint operator from $\mathfrak{S}_\infty$.

We may suppose without loss of generality that the operator $I-T$ is invertible. In fact, replacing the parameter λ by $\lambda+a$ in the bundle (8.7) obviously does not alter its eigenvectors and associated vectors, and only shifts the spectrum of characteristic numbers of the bundle by a. Under this shift, the bundle (8.7) goes over into the bundle

$$L(\lambda+a)=I-T-aH-\lambda H. \tag{8.9}$$

On the other hand, if we choose a such that $|(I-aH)^{-1}T|<1$, the operator

$$I-T-aH=(I-aH)[I-(I-aH)^{-1}T]$$

will be invertible.

Thus, assuming the existence of the bounded operator $(I - T)^{-1}$, we can write

$$(I - T)^{-1} = I + S,$$

where S is some completely continuous operator.

It is obvious that multiplying any bundle $L(\lambda)$ from the left by a continuous and invertible operator U does not change its eigenvectors and associated vectors. We multiply the bundle (8.7) from the left by the operator $U = (I - T)^{-1}$; it then goes over into the bundle $L_1(\lambda) = I - \lambda A_1$, where $A_1 = (I + S)H$.

Let λ_0 be a characteristic number of the bundle $L_1(\lambda)$, and $\phi_0, \phi_1, \cdots, \phi_m$ some chain consisting of an eigenvector and associated vectors, corresponding to the number λ_0, i.e.

$$(I - \lambda_0 A_1)\phi_0 = 0, \quad (I - \lambda_0 A_1)\phi_k = A_1\phi_{k-1} \qquad (k = 1, 2, \cdots, m).$$

The first relation shows that ϕ_0 is an eigenvector of the operator A_1, and the others imply the equality

$$(I - \lambda_0 A_1)^{k+1}\phi_k = A_1(I - \lambda_0 A_1)^k\phi_{k-1} = \cdots = A_1^k(I - \lambda_0 A_1)\phi_0 = 0.$$

Thus $\phi_1, \cdots, \phi_m$ are root vectors of A_1.

Conversely, if ϕ is some root vector of order k of A_1, corresponding to the eigenvalue λ_0, i.e., if $(I - \lambda_0 A_1)^k\phi \neq 0$, $(I - \phi_0 A_1)^{k+1}\phi = 0$, then, setting $\phi_k = \phi$,

$$\phi_{k-1} = -\sum_{p=1}^{k+1}\binom{k+1}{p}(-\lambda_0)^p A_1^{p-1}\phi_k - \lambda_0\phi_k,$$

we will have

$$(I - \lambda_0 A_1)\phi_k - A_1\phi_{k-1} = (I - \lambda_0 A_1)^{k+1}\phi_k = 0,$$

and consequently

$$A_1(I - \lambda_0 A_1)^k\phi_{k-1} = (I - \lambda_0 A_1)^{k+1}\phi_k = 0,$$

$$(I - \lambda_0 A_1)^k\phi_{k-1} = 0.$$

Similarly, from ϕ_{k-1} we can construct the vector ϕ_{k-2}, and then from ϕ_{k-2} the vector ϕ_{k-3} and so on, until an entire chain of vectors $\phi_k, \phi_{k-1}, \cdots, \phi_0$ $(\neq 0)$, satisfying the relations (8.8),[17] has been reconstructed.

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[17] We remark that the chain $\phi_0, \phi_1, \cdots, \phi_k$ is a Jordan chain for the unbounded operator $B = A_1^{-1} = H^{-1}(I - T)$.

We have thus proved that the characteristic numbers of the bundle (8.7) coincide with the characteristic numbers of the completely continuous operator A_1, and the linear hull of the eigenvectors and associated vectors of the bundle (8.7), corresponding to any characteristic number λ_0 of this bundle, coincides with the root subspace of the operator A_1 corresponding to the same characteristic number. The dimension of this hull (root subspace) is called the *algebraic multiplicity* of the characteristic number λ_0 of the bundle (8.7).

It follows (independently of whether or not the operator $I - T$ has a continuous inverse) that the spectrum of characteristic numbers of the bundle (8.7) is *discreet*, i.e. every characteristic number has finite multiplicity and the only possible limit point of the spectrum of all these numbers is the point ∞.

At the same time, we have shown that under the condition that the operator $I - T$ is invertible, the completeness of the system of root vectors of the operator $A_1 = (I + S) H$, where $(I - T)^{-1} = I + S$, is the same as the completeness of the system of eigenvectors and associated vectors of the linear bundle $L(\lambda) = I - T - \lambda H$. Here the operator A_1 can be replaced by the similar operator $A = H(I + S) = (I + S)^{-1} A_1 (I + S)$.

3. By analyzing the proof of Theorem 8.1 of M. V. Keldyš, it is easily discovered that Lemma 8.1 on the connection between the components of a Volterra operator makes it possible to weaken the conditions in the theorem. Namely, Theorem 8.1 remains valid if, in its statement, the condition $p(H) < \infty$ is replaced by the weaker condition $p([HS]_{\mathscr{I}}) < \infty$ $([HS]_{\mathscr{I}} = (HS - S^*H)/2i)$. With this replacement the original proof remains valid, if it is shown *in addition* that the Volterra operator $A_1 = QH(I + S) Q$ which appears in the proof has finite order.

The imaginary component of the operator A_1 is obviously the operator $Q(HS - S^*H) Q/2i$, which belongs to $\mathfrak{S}_p$ for some $p < \infty$. Since A_1 is a Volterra operator, by Lemma 8.1 $p(A_1) < \infty$.

Theorem 8.2 for a linear bundle admits a similar generalization.

THEOREM 8.3. *Let $H \in \mathfrak{S}_\infty$ be a complete selfadjoint operator, and let $T \in \mathfrak{S}_\infty$. Then the system of eigenvectors and associated vectors of the linear bundle $I - T - \lambda H$ is complete in $\mathfrak{H}$ whenever $p([HT]_{\mathscr{I}}) < \infty$.*

PROOF. We note that for the shifted bundle $L_1(\lambda) = L(\lambda + a) = I - T_1 - \lambda H$ the operator T_1 equals $T + aH$, and consequently $T_1 H - HT_1^*$

$= TH - HT^*$, if a is real. Therefore we may assume without loss of generality that the operator $I - T$ is invertible, and in this case the completeness of the system of eigenvectors and associated vectors of the bundle is equivalent to the completeness of the root vectors of the operator $A = H(I-T)^{-1} = H(I+S)$. If $HT - T^*H \in \mathfrak{S}_p$ for some $p > 0$, then $H(I-T) - (I-T^*)H \in \mathfrak{S}_p$, and consequently $(I-T^*)^{-1}H - H(I-T)^{-1} \in \mathfrak{S}_p$. Hence

$$\begin{aligned} HS - S^*H &= H(I+S) - (I+S^*)H \\ &= H(I-T)^{-1} - (I-T^*)^{-1}H \in \mathfrak{S}_p. \end{aligned}$$

But under this condition the completeness of the system of root vectors of the operator A has already been established.

We remark, in conclusion, that V. I. Macaev [2] has shown by other means that Theorem 8.1 remains valid if in its formulation the condition $p(H) < \infty$ is replaced by the condition $S \in \mathfrak{S}_\omega$. Therefore Theorem 8.2 remains valid if the condition $p(H) < \infty$ in it is replaced by the condition $T \in \mathfrak{S}_\omega$. In Macaev's communication [5] one can find a number of other generalizations of the theorems of M. V. Keldyš.[18]

4. In this case where both H and S are selfadjoint operators, Theorem 8.1 can be strengthened:

1. *The system of root vectors of the operator $A = H(I+S)$ is complete in $\mathfrak{H}$ whenever H and S are completely continuous selfadjoint operators and the operator A vanishes only at zero.*

We present a proof of this result. Under the specified conditions, the operators H and $I+S$ are obviously complete.

We introduce a new (possibly indefinite) scalar product in $\mathfrak{H}$, setting

$$\{f,g\} = ((I+S)f,g) = (Jf,g) \qquad (J = I+S). \tag{8.10}$$

With respect to this inner product the operator A will be selfadjoint:

$$\{Af,g\} = \{f,Ag\} \qquad (f,g \in \mathfrak{H}).$$

If the operator J is positive[19] (all eigenvalues of S are greater than -1), then (and only then) the product $\{f,g\}$ will be definite, i.e.

[18] In a recent communication A. S. Markus [5] presented the generalization of the theorems of M. V. Keldyš to the case of a Banach space. This note and a paper by Ju. I. Ljubič [2] also contain other tests for completeness for operators in Banach spaces.

[19] For the bundle (8.7), to this case will correspond the case in which the operator $I - T$ $(= (I+S)^{-1})$ is positive.

$\{f,f\} > 0$ for $f \neq 0$. In this case the norms $|f| = \sqrt{(f,f)}$ and $|f|_1 = \sqrt{\{f,f\}}$ are topologically equivalent.

Taking (8.10) into consideration, we conclude that in this case the spectrum of the operator A will be real and that A will have a complete in $\mathfrak{H}$ and indeed *J-orthonormal* system of eigenvectors $\{e_j\}_1^\infty$, i.e.,

$$(Je_j, e_k) = \{e_j, e_k\} = \delta_{jk} \qquad (j,k = 1,2,\cdots).$$

The system $\{e_j\}_1^\infty$ will form a Riesz basis (see the definition in §2, Chapter VI).

Let us consider the second possibility, where S has eigenvalues < -1, and hence the scalar product $\{f,g\}$ is indefinite.[20] In this case the space $\mathfrak{H}$ can be broken up into the orthogonal sum

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$$

of subspaces which are invariant with respect to S, where $\mathfrak{H}_1$ is the finite-dimensional subspace in which the eigenvalues of S are less than -1, and $\mathfrak{H}_2$ is the infinite-dimensional space in which the eigenvalues of S are greater that -1.

It is obvious that the form $\{f,f\}$ will be negative on $\mathfrak{H}_1$ and positive on $\mathfrak{H}_2$; moreover, the norms $|f|$ and $|f|_1$ are topologically equivalent on $\mathfrak{H}_2$.

Thus the space $\mathfrak{H}$ with the indefinite scalar product $\{f,g\}$ can be regarded as a Pontrjagin space Π_κ, where κ is the dimension of $\mathfrak{H}_1$ (see Pontrjagin [**1**] and Iohvidov and Kreĭn [**1**]).

Since A is a completely continuous selfadjoint operator in Π_κ which vanishes only at zero, by a theorem of I. S. Iohvidov [**1**] the system of root vectors of A is complete in $\mathfrak{H}$.

This result, combined with results of Pontrjagin [**1**], also enables us to state the following result.

2. *Under the hypotheses of result* 1, *the space* $\mathfrak{H}$ *can be broken up into the direct sum*

$$\mathfrak{H} = \mathfrak{L} + \mathfrak{N}$$

of two subspaces which are invariant with respect to the operator A, *where* $\mathfrak{L}$ *is not more that* 2κ*-dimensional, and the quadratic form* $\{f,f\}$ *is positive on* $\mathfrak{N}$, *and such that the operator* A *has in* $\mathfrak{N}$ *a complete J-orthonormal system of eigenvectors* $\{e_j\}_1^\infty$, *to which correspond real eigenvalues.*

[20] The point $\lambda = -1$ is regular for the operator S, since $I + S$ vanishes only at zero and $S \in \mathfrak{S}_\infty$.

The form $\{f,f\}$ will be nondegenerate on $\mathfrak{L}$ and will have precisely κ negative squares. There are infinitely many ways of constructing a J-orthonormal basis in $\mathfrak{L}$ which, together with the basis $\{e_j\}$ of the subspace $\mathfrak{N}$, forms a Riesz basis (see §2, Chapter VI) of the entire space $\mathfrak{H}$.

The structure of the root subspaces of the operator A in $\mathfrak{L}$ will be determined by the well-known results on the structure of the root lineals of a linear operator, acting in a finite-dimensional space with an indefinite metric with respect to which it is selfadjoint (cf. Pontrjagin [1], Iohvidov and Kreĭn [1] and Mal′cev [1]). In particular $\mathfrak{L}$ will contain all root lineals of the operator A corresponding to nonreal eigenvalues.

It is easily shown that if H is a positive operator, then the subspace $\mathfrak{L}$ consists only of the null vector.

§9. Theorems on the multiple completeness of the system of eigenvectors and associated vectors of an operator bundle

1. The question of multiple completeness was raised by M. V. Keldyš [1] in connection with the consideration, for vector-functions $x(t)$ with values in $\mathfrak{H}$, of an operator differential equation of the form

$$\left(I - A_0 - A_1\frac{d}{dt} - \cdots - A_n\frac{d^n}{dt^n}\right) x = 0, \tag{9.1}$$

where the A_j $(j = 1,2,\cdots,n)$ are operators from $\mathfrak{S}_\infty$.

As in the case of equations of the type (9.1) in finite-dimensional spaces, the general theory of equations of the type (9.1) is based on the theory of operator bundles:

$$L(\lambda) = I - A_0 - \lambda A_1 - \cdots - \lambda^n A_n.$$

If we seek a solution of the equation (9.1) in the form $x(t) = e^{\lambda_0 t}\phi$, where the vector ϕ from $\mathfrak{H}$ does not depend on t, then we are naturally led to the equation

$$L(\lambda_0)\,\phi = 0.$$

The number λ_0 is called a *characteristic number* of the bundle $L(\lambda)$, if the equation $L(\lambda_0)\,\phi = 0$ has a nontrivial solution ϕ $(\neq 0)$. This solution is called an *eigenvector* of the bundle $L(\lambda)$.

If some function $x(t)$ of the form

$$x(t) = e^{\lambda_0 t}\left(\frac{t^k}{k!}\,\phi_0 + \frac{t^{k-1}}{(k-1)!}\,\phi_1 + \cdots + \frac{t}{1!}\,\phi_{k-1} + \phi_k\right), \tag{9.2}$$

where $\phi_j \in \mathfrak{H}$ $(j = 0, 1, \cdots, k; \phi_0 \neq 0)$, is a solution of the equation (9.1), then the functions

$$\left(\frac{d}{dt} - \lambda_0 I\right) x = e^{\lambda_0 t}\left(\frac{t^{k-1}}{(k-1)!}\phi_0 + \cdots + \frac{t}{1!}\phi_{k-2} + \phi_{k-1}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\frac{d}{dt} - \lambda_0 I\right)^k x = e^{\lambda_0 t}\phi_0.$$

will also be solutions of this equation. It follows that ϕ_0 is an eigenvector of the bundle $L(\lambda)$, corresponding to the value λ_0.

It is easily seen that a function $x(t)$ of the form (9.2) will be a solution of the equation (9.1) if and only if

$$L(\lambda_0)\,\phi_p + \frac{1}{1!}\,\frac{\partial L(\lambda_0)}{\partial \lambda_0}\,\phi_{p-1} + \cdots + \frac{1}{p!}\,\frac{\partial L(\lambda_0)}{\partial \lambda_0^p}\,\phi_0 = 0 \qquad (p = 0, 1, \cdots, k). \tag{9.3}$$

The vectors $\phi_1, \phi_2, \cdots, \phi_k$ are said to be *associated* with the eigenvector ϕ_0. The number $k + 1$ is called the *length* of the chain $\phi_0, \phi_1, \cdots, \phi_k$; it can be finite or infinite. The vector ϕ_0 is said to be an eigenvector of the bundle of *finite rank* r if the longest chain corresponding to the vector ϕ_0 has length r.

A characteristic number λ_0 of the bundle $L(\lambda)$ is said to be a characteristic number of *finite algebraic multiplicity* if the subspace $\mathfrak{Z}(L(\lambda_0))$ (of zeros of the operator $L(\lambda_0)$) is finite-dimensional and the ranks of all the eigenvectors $\phi \in \mathfrak{Z}(L(\lambda_0))$ have a common bound.[21)]

It is obvious that in passing from the bundle $L(\lambda)$ to the bundle $L_1(\lambda) = L(\lambda + a)$ (a any complex number) the spectrum of the characteristic numbers of $L_1(\lambda)$ is obtained from the spectrum of $L(\lambda)$ by the shift $\lambda \to \lambda - a$, and the eigenvectors and associated vectors remain the same.

We remark that the method for proving the results presented below consists in reducing a polynomial operator bundle to a linear bundle.

1. *If for at least one point* $\lambda = \lambda_0$ *the operator*

[21)] As will be made clear from what follows, one has to understand, by the algebraic multiplicity of the characteristic number λ_0 of the bundle $L(\lambda)$, the multiplicity of the characteristic number λ_0 of the operator $\widetilde{L}_1(1 - \widetilde{L}_0)^{-1}$, which is constructed from the bundle $L(\lambda)$ by a method which will be indicated further on.

$$L(\lambda_0) = I - \sum_{j=0}^{n} \lambda_0^j A_j \qquad (A_j \in \mathfrak{S}_\infty)$$

is invertible, then the set of characteristic numbers of the bundle $L(\lambda)$ *consists of isolated points of finite algebraic multiplicity.* [22]

That the characteristic numbers form an isolated set of points follows at once from Theorem I.5.1. In fact, $L(\lambda) - I$ is an entire operator-function taking values in $\mathfrak{S}_\infty$, and the operator $L(\lambda)$ is invertible; consequently by Theorem I.5.1 the operator $L(\lambda)$ is invertible everywhere with the possible exception of a set of isolated points, which are the characteristic numbers of the bundle $L(\lambda)$.

To prove that every characteristic number of the bundle has finite algebraic multiplicity, we rewrite (9.1) as a first-order equation. To this end we form the Hilbert space $\tilde{\mathfrak{H}}$, the orthogonal sum of n copies of the space $\mathfrak{H}$. Then the equation

$$L(\lambda) g = f \tag{9.4}$$

can be written in the form of a system

$$\begin{aligned} g^{(0)} - A_0 g^{(0)} - \cdots - A_{n-1} g^{(n-1)} - \lambda A_n g^{(n-1)} &= f, \\ g^{(1)} - \lambda g^{(0)} &= 0, \\ g^{(2)} - \lambda g^{(1)} &= 0, \\ \cdots\cdots\cdots\cdots & \\ g^{(n-1)} - \lambda g^{(n-2)} &= 0, \end{aligned}$$

where $g^{(0)} = g$. Thus it is equivalent to the following equation in the space $\tilde{\mathfrak{H}}$:

$$\tilde{L}(\lambda)\tilde{g} = (\tilde{I} - \tilde{L}_0)\tilde{g} - \lambda \tilde{L}_1 \tilde{g} = \tilde{f}, \tag{9.5}$$

where $\tilde{g} = \{g^{(j)}\}_0^{n-1} \in \tilde{\mathfrak{H}}$, $\tilde{f} = \{f, 0, \cdots 0\} \in \tilde{\mathfrak{H}}$, and the operators $\tilde{L}_1$ and $\tilde{L}_0$ are defined in $\tilde{\mathfrak{H}}$ by the matrices

$$\tilde{L}_0 = \begin{pmatrix} A_0 & A_1 \cdots & A_{n-1} \\ 0 & 0 \cdots & 0 \\ \cdot & \cdots & \cdot \\ 0 & 0 \cdots & 0 \end{pmatrix}, \qquad \tilde{L}_1 = \begin{pmatrix} 0 & 0 \cdots 0 & A_n \\ I & 0 \cdots 0 & 0 \\ 0 & I \cdots 0 & 0 \\ \cdot & \cdots & \cdot \\ 0 & 0 \cdots I & 0 \end{pmatrix}.$$

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[22] A more general result was proved by A. S. Markus [1].

From the equivalence of the equations (9.4) and (9.5) it follows at once that the linear bundle $\widetilde{L}(\lambda)$ has the same characteristic numbers as the bundle $L(\lambda)$.

Moreover it is easily shown that the vector $\phi_0 = \{\phi_0^{(j)}\}_0^{n-1}$ $(\in \widetilde{\mathfrak{H}})$ will be an eigenvector of the bundle $\widetilde{L}(\lambda)$, corresponding to the characteristic number λ_0, and the vectors $\phi_p = \{\phi_p^{(j)}\}_0^{n-1}$ $(\in \widetilde{\mathfrak{H}}; p = 1, 2, \cdots, k)$ will be associated with ϕ_0 if and only if $\phi_0^{(0)}$ is an eigenvector of the bundle $L(\lambda)$ corresponding to the characteristic number λ_0, the vectors $\phi_p^{(0)}$ $(p = 1, 2, \cdots, k)$ are associated with $\phi_0^{(0)}$, and

$$\phi_p^{(j)} = \frac{d^j}{dt^j} e^{\lambda_0 t} \left(\phi_p^{(0)} + \phi_{p-1}^{(0)} \frac{t}{1!} + \cdots + \phi_0^{(0)} \frac{t^p}{p!} \right) \Bigg|_{t=0}$$

$$(j = 1, 2, \cdots, n-1; p = 1, 2, \cdots, k).$$

It follows in particular that the lengths of corresponding chains of associated vectors of the bundles $\widetilde{L}(\lambda)$ and $L(\lambda)$ coincide.

We may assume without loss of generality that the operator $\widetilde{I} - \widetilde{L}_0$ is invertible (in the contrary case we could go over from the bundle $\widetilde{L}(\lambda)$ to the bundle $\widetilde{L}(\lambda + \lambda_0)$). Therefore by virtue of §8.2 (see p. 261) it remains to show that all the nonzero points of the spectrum of the operator $\widetilde{L}_1(\widetilde{I} - \widetilde{L}_0)^{-1}$ are normal eigenvalues.

We already know that all the nonzero points of the spectrum of this operator are isolated points. Since

$$\widetilde{L}_1^n = \begin{pmatrix} A_n & 0 & \cdots & 0 \\ 0 & A_n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A_n \end{pmatrix} = \| \delta_{jk} A_n \|_1^n,$$

it follows that $\widetilde{L}_1^n \in \mathfrak{S}_\infty$, and consequently all of its nonzero spectrum consists of normal eigenvalues. But then the nonzero spectrum of $\widetilde{L}_1$ consists of normal eigenvalues.[23)]

Since $\widetilde{L}_1(\widetilde{I} - \widetilde{L}_0)^{-1} = \widetilde{L}_1 T + \widetilde{L}_1$, where $T = (\widetilde{I} - \widetilde{L}_0)^{-1} - \widetilde{I} \in \mathfrak{S}_\infty$, according to Lemma I.5.2 all points λ $(\neq 0)$ are normal points of the operator $\widetilde{L}_1(\widetilde{I} - \widetilde{L}_0)^{-1}$. The result is proved.

[23)] We have used here the following elementary result, which the reader can easily establish for himself. Let λ_0 be an eigenvalue of the operator A, and hence λ_0^n an eigenvalue of the operator A^n. If λ_0^n is a normal eigenvalue of A^n, then λ_0 is a normal eigenvalue of A.

2. Let us associate, with every characteristic number λ_0 of the bundle $L(\lambda)$ and every system $\phi_0, \phi_1, \cdots, \phi_k$ consisting of a corresponding eigenvector and associated vectors, a system of nk vectors $\phi_p^{(j)}$, constructed according to the rules

$$\phi_p^{(0)} = \phi_p \qquad (p = 1, 2, \cdots, k)$$

and

$$\phi_p^{(j)} = \frac{d^j}{dt^j} e^{\lambda_0 t} \left(\phi_p^{(0)} + \phi_{p-1}^{(0)} \frac{t}{1!} + \cdots + \frac{t^p}{p!} \phi_0^{(0)} \right) \Bigg|_{t=0}$$

$$(j = 1, 2, \cdots, n-1; p = 1, 2, \cdots, k).$$

Following M. V. Keldyš [1], we shall say that the system of all eigenvectors and associated vectors of the bundle $L(\lambda)$ is *n-fold complete* if the union of all systems of the form $\{\phi_p^{(j)}\}_0^{n-1}$ $(p = 1, 2, \cdots, k)$ forms a complete system in the space $\widetilde{\mathfrak{H}}$ consisting of the orthogonal sum of n copies of the space $\mathfrak{H}$. For the case $n = 1$ this concept reduces to the concept of completeness in the usual sense.

Recalling the proof of result 1, we conclude that the system of eigenvectors and associated vectors of the bundle $L(\lambda)$ is n-fold complete if and only if the system of eigenvectors and associated vectors of the linear bundle $\widetilde{L}(\lambda)$ is complete.

We further note that for a differential equation (9.1) the n-fold completeness of the system of eigenvectors and associated vectors of the bundle $L(\lambda)$ means that there exists a solution $x(t)$ of the equation (9.1) which is a linear combination of solutions of the form (9.2) and whose initial values $x(0), x'(0), \cdots, x^{(n-1)}(0)$ are arbitrarily close to any preassigned values.

THEOREM 9.1 (M. V. KELDYŠ [1][24)]). *Let H be any complete normal operator of finite order, and suppose that for some positive integer n the operator H^n is selfadjoint. Then the system of eigenvectors and associated vectors of each of the two conjugate bundles*

[24)] In this paper the theorem is formulated under more restrictive assumptions concerning the bundle $K(\lambda)$, namely

$$K(\lambda) = I - B_0 - \lambda H_1 B_1 - \cdots - \lambda^{n-1} H_1 B_{n-1} - \lambda^n H_1,$$

where $B_0 \in \mathfrak{S}_\infty$, $B_j \in \mathfrak{R}$ $(j = 1, 2, \cdots, n-1)$, and H_1 is a complete selfadjoint operator of finite order.

In the form which we have given, the theorem was announced in a paper by D. È. Allahverdiev [1].

(9.6) $$K(\lambda) = I - T_0 - \lambda H T_1 - \cdots - \lambda^{n-1} H^{n-1} T_{n-1} - \lambda^n H^n$$

and

(9.7) $$K^*(\lambda) = I - T_0^* - \lambda T_1^* H - \cdots - \lambda^{n-1} T_{n-1}^* H^{n-1} - \lambda^n H^n,$$

where $T_j \in \mathfrak{S}_\infty$ $(j = 0, 1, \cdots, n-1)$, *is n-fold complete in* $\mathfrak{H}$. *For any* $\epsilon > 0$, *all the characteristic numbers of the bundle* $K(\lambda)$, *with the possible exception of a finite number of them, lie in the sectors*

$$\frac{\pi k}{n} - \epsilon < \arg \zeta < \frac{\pi k}{n} + \epsilon \qquad (k = 0, 1, \cdots, 2n-1).$$

We precede the proof of this theorem by the following lemma.

LEMMA 9.1. *Let* $H \in \mathfrak{S}_\infty$ *be a complete normal operator whose nth power* H^n *is a selfadjoint operator, and let* $T_j \in \mathfrak{S}_\infty$ $(j = 0, 1, \cdots, n-1)$. *Then for any* ϵ $(0 < \epsilon < \pi/2n)$ *there exists* $\rho \geqq 0$ *such that at all points* λ *with* $|\lambda| \geqq \rho$ *of the region* F_ϵ, *obtained from the complex plane by removing the* $2n$ *sectors* [25]

(9.8) $$|\arg z - \pi k/n| < \epsilon \qquad (k = 0, 1, \cdots, 2n-1),$$

the operator bundle

$$K(\lambda) = I - T_0 - \lambda H T_1 - \cdots - \lambda^{n-1} H^{n-1} T_{n-1} - \lambda^n H^n$$

is invertible, and

$$\sup_{\lambda \in F_\epsilon;\, |\lambda| \geqq \rho} |K^{-1}(\lambda)| < \infty.$$

PROOF. We associate with the bundle $K(\lambda)$ the linear bundle $\widetilde{K}_1(\lambda) = \widetilde{I} - \widetilde{T} - \lambda \widetilde{H}$ with operators $\widetilde{T}$ and $\widetilde{H}$ which are defined in the orthogonal sum $\widetilde{\mathfrak{H}}$ of n copies of $\mathfrak{H}$ by the matrices

$$\widetilde{T} = \begin{pmatrix} T_0 & 0 \cdots 0 \\ T_1 & 0 \cdots 0 \\ \cdot\ \cdot & \cdot\ \cdot \\ T_n & 0 \cdots 0 \end{pmatrix}; \qquad \widetilde{H} = \begin{pmatrix} 0 & H \cdots 0 & 0 \\ 0 & 0 \cdots 0 & 0 \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 \cdots 0 & H \\ H & 0 \cdots 0 & 0 \end{pmatrix}.$$

The operator $\widetilde{T}$ is completely continuous, and the operator $\widetilde{H}$ is

[25] If the operator H^n has a finite number of negative eigenvalues, then the sectors (9.8) can be replaced by the following n sectors:

$$|\arg z - 2\pi k/n| < \epsilon \qquad (k = 0, 1, \cdots, n-1).$$

a complete, completely continuous normal operator. Since $\widetilde{H}^n = \| \delta_{jk} H^n \|_1^n$ is selfadjoint, it follows that all the characteristic numbers of the operator $\widetilde{H}$ lie on the rays $\arg \lambda = \pi k/n$ $(k = 0, 1, \cdots, 2n-1)$, and consequently at all points λ $(\neq 0)$ of the region F_ϵ we will have, for the operator $(\widetilde{I} - \lambda \widetilde{H})^{-1} = -\lambda^{-1}(\widetilde{H} - \lambda^{-1}\widetilde{I})^{-1}$,

$$|(\widetilde{I} - \lambda\widetilde{H})^{-1}| \leqq |\lambda^{-1}|/d(1/\lambda) \leqq 1/\sin\epsilon, \tag{9.9}$$

where $d(1/\lambda)$ is the distance from the point λ^{-1} to the closest of the rays $\arg z = \pi k/n$ $(k = 0, 1, \cdots, 2n-1)$.

According to Lemma 7.1,

$$\lim_{\lambda \in F_\epsilon; \lambda \to \infty} |\widetilde{T}(\widetilde{I} - \lambda\widetilde{H})^{-1}| = 0.$$

We chose a number $\rho > 0$ such that

$$|\widetilde{T}(\widetilde{I} - \lambda\widetilde{H})^{-1}| = q < 1 \quad \text{for} \quad |\lambda| \geqq \rho, \ \lambda \in F_\epsilon.$$

Then at all points λ of the region F_ϵ for which $|\lambda| \geqq \rho$, the operator $\widetilde{I} - \widetilde{T}(\widetilde{I} - \lambda\widetilde{H})^{-1}$ will be invertible, and

$$\sup_{\lambda \in F_\epsilon; |\lambda| \geqq \rho} |(\widetilde{I} - \widetilde{T}(\widetilde{I} - \lambda\widetilde{H})^{-1})^{-1}| \leqq \frac{1}{1-q} \quad (< \infty). \tag{9.10}$$

Since the operator $\widetilde{K}_1(\lambda)$ can be represented in the form $\widetilde{K}_1(\lambda) = (\widetilde{I} - \widetilde{T}(\widetilde{I} - \lambda\widetilde{H})^{-1})(\widetilde{I} - \lambda\widetilde{H})$, it is invertible at all points $\lambda \in F_\epsilon$ with $|\lambda| \geqq \rho$, and

$$\widetilde{K}_1^{-1}(\lambda) = (\widetilde{I} - \lambda\widetilde{H})^{-1}(\widetilde{I} - \widetilde{T}(\widetilde{I} - \lambda\widetilde{H})^{-1})^{-1}.$$

It follows by virtue of (9.9) and (9.10) that

$$\sup_{\lambda \in F_\epsilon; |\lambda| \geqq \rho} |\widetilde{K}_1^{-1}(\lambda)| < \infty.$$

The equation

$$K(\lambda)g = f \tag{9.11}$$

is obviously equivalent to the system of equations

$$\begin{gathered} g^{(0)} - T_0 g^{(0)} - \lambda H g^{(1)} = f, \\ g^{(1)} - T_1 g^{(0)} - \lambda H g^{(2)} = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ g^{(n-1)} - T_{n-1} g^{(0)} - \lambda H g^{(0)} = 0, \end{gathered} \tag{9.12}$$

where $g = g^{(0)}$, or, what is the same, to the equation

$$\widetilde{K}_1(\lambda)\widetilde{g} = \widetilde{f}, \tag{9.13}$$

where $\tilde{g} = \{g^{(j)}\}_0^{n-1}$, $\tilde{f} = \{f, 0, \cdots, 0\} \in \tilde{\mathfrak{H}}$. It follows in particular that the bundles $K(\lambda)$ and $\tilde{K}_1(\lambda)$ have the same characteristic numbers.

If λ is not a characteristic number of the bundle $K(\lambda)$, then the solution g of the equation (9.11) equals the first coordinate of the vector $\tilde{K}_1^{-1}(\lambda)\tilde{f}$, and so

$$|g| = |K^{-1}(\lambda) f| \leqq |\tilde{K}_1^{-1}(\lambda)\tilde{f}| \leqq |\tilde{K}_1^{-1}(\lambda)|\,|f|.$$

Thus

$$|K^{-1}(\lambda)| \leqq |\tilde{K}_1^{-1}(\lambda)| \qquad (|\lambda| \geqq \rho;\ \lambda \in F_\epsilon).$$

The lemma is proved.

PROOF OF THEOREM 9.1.[26] The last assertion of the theorem is an immediate consequence of Lemma 9.1.

We may assume without loss of generality that the operator $I - T_0$ is invertible (in the contrary case we could carry out the parameter shift $\lambda \to \lambda + a$ on the bundle $L(\lambda)$, with a an appropriate number).

As was already noted, the n-fold completeness of the system of eigenvectors and associated vectors of the bundle $K(\lambda)$ coincides with the ordinary completeness of the system of eigenvectors and associated vectors of the linear bundle $\tilde{K}(\lambda) = \tilde{I} - \tilde{K}_0 - \lambda\tilde{K}_1$, where the operators $\tilde{K}_0$ and $\tilde{K}_1$ are defined in $\tilde{\mathfrak{H}}$ by the matrices

$$\tilde{K}_0 = \begin{pmatrix} T_0 & HT_1 & \cdots & H^{n-1}T_{n-1} \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{K}_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & H^n \\ I & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}.$$

Since by hypothesis the operator $I - T_0$ is invertible, the operator $\tilde{I} - \tilde{K}_0$ is also invertible. Consequently the problem reduces to proving the completeness of the system of root vectors of the operator $\tilde{A} = (\tilde{I} + \tilde{S})\tilde{K}_1$, where $\tilde{I} + \tilde{S} = (\tilde{I} - \tilde{K}_0)^{-1}$.

Let us denote by $\tilde{\mathfrak{H}}_0$ the orthogonal complement of the linear hull of all the root vectors of the operator $\tilde{A}$. According to Remark I.4.2, the spectrum of the operator $\tilde{A}_1 = \tilde{Q}\tilde{A}\tilde{Q}$, where $\tilde{Q}$ is the orthoprojector which projects $\tilde{\mathfrak{H}}$ onto $\tilde{\mathfrak{H}}_0$, is concentrated at zero.

Since the subspace $\tilde{\mathfrak{H}}_0^\perp$ is invariant with respect to $\tilde{A}$, for any point λ at which the operator $\tilde{I} - \lambda\tilde{A}$ is invertible one has, by result 2, §1, Chapter I, the relation

[26] This proof uses certain arguments from a note by Ju. A. Palant [1].

$$(9.14) \qquad (\tilde{I} - \lambda\tilde{A}_1)^{-1} = \tilde{Q}(\tilde{I} - \lambda\tilde{A})^{-1}\tilde{Q} + \tilde{P},$$

where $\tilde{P} = \tilde{I} - \tilde{Q}$. The operator-function $R(\lambda) = (\tilde{I} - \lambda\tilde{A}_1)^{-1}$ is obviously entire. We shall show that in the region F_ϵ, obtained from the entire complex plane by removing the sectors (9.8), one has the relation

$$(9.15) \qquad |R(\lambda)| = O(|\lambda^{2n-2}|) \qquad (\lambda \to \infty,\ \lambda \in F_\epsilon).$$

Taking into consideration the equalities (9.14) and the relation

$$(\tilde{I} - \lambda\tilde{A})^{-1} = (\tilde{I} - \lambda(\tilde{I} - \tilde{K}_0)^{-1}\tilde{K}_1)^{-1} = (\tilde{I} - \tilde{K}_0 - \lambda\tilde{K}_1)^{-1}(\tilde{I} - \tilde{K}_0),$$

we obtain

$$|R(\lambda)| \leqq |\tilde{I} - \tilde{K}_0|\ |\tilde{K}^{-1}(\lambda)| + 1.$$

Thus (9.15) will be established once we have shown that

$$(9.16) \qquad |\tilde{K}^{-1}(\lambda)| = O(|\lambda^{2n-2}|) \qquad (\lambda \in F_\epsilon,\ \lambda \to \infty).$$

It is not difficult to see that the operator $\tilde{K}^{-1}(\lambda)$ is defined in $\tilde{\mathfrak{H}}$ by the matrix

$$\tilde{K}^{-1}(\lambda) = \begin{pmatrix} K^{-1}(\lambda) & K^{-1}(\lambda)M_1 & \cdots & K^{-1}(\lambda)M_{n-1} \\ \lambda K^{-1}(\lambda) & I + \lambda K^{-1}(\lambda)M_1 & \cdots & \lambda K^{-1}(\lambda)M_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda^{n-1}K^{-1}(\lambda) & \lambda^{n-2}I + \lambda^{n-1}K^{-1}(\lambda)M_1 & \cdots & I + \lambda^{n-1}K^{-1}(\lambda)M_{n-1} \end{pmatrix},$$

where

$$M_j = H^jT_j + \lambda H^{j+1}T_{j+1} + \cdots + \lambda^{n-j-1}H^{n-1}T_{n-1} + \lambda^{n-j}H^n$$
$$(j = 1, 2, \cdots, n-1).$$

According to Lemma 9.1,

$$K^{-1}(\lambda) = O(1) \qquad (\lambda \to \infty,\ \lambda \in F_\epsilon).$$

Consequently it follows from (9.17) that the relation (9.16) holds, and with it the relation (9.15).

The operator-function $R(\lambda)$ is an entire operator-function of order $\leqq np$ $(p > p(H))$. In fact, since $H \in \mathfrak{S}_p$, the operator

$$\tilde{K}_0\tilde{K}_1 = \begin{pmatrix} HT_1 & H^2T_2 \cdots H^{n-1}T_{n-1} & T_0H^n \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots 0 & 0 \\ 0 & 0 \cdots 0 & 0 \end{pmatrix}$$

belongs to $\mathfrak{S}_p$, and with it the operator $\widetilde{S}\widetilde{K}_1 = (\widetilde{I} - \widetilde{K}_0)^{-1}\widetilde{K}_0\widetilde{K}_1$ belongs to $\mathfrak{S}_p$. Therefore

$$\widetilde{A}_1^n = \widetilde{Q}\widetilde{A}^n\widetilde{Q} = \widetilde{Q}(\widetilde{K}_1 + \widetilde{S}\widetilde{K}_1)^n\widetilde{Q} = \widetilde{Q}\widetilde{K}_1^n\widetilde{Q} + \widetilde{Q}M\widetilde{Q},$$

where $\widetilde{M} \in \mathfrak{S}_p$. Since the operator $\widetilde{K}_1^n = \| \delta_{jk}H^n \|_1^n$ also belongs to $\mathfrak{S}_p$, it follows that $\widetilde{A}_1^n \in \mathfrak{S}_p$.

Taking Theorem 5.2 into account, we can now conclude that the operator-function $(\widetilde{I} - \lambda^n\widetilde{A}_1^n)^{-1}$ is an entire operator-function of order $\leq np$. But it then follows from

$$R(\lambda) = (\widetilde{I} + \lambda\widetilde{A}_1 + \cdots + \lambda^{n-1}\widetilde{A}_1^{n-1})(\widetilde{I} - \lambda^n\widetilde{A}_1^n)^{-1}$$

that the same will be true of the entire operator-function $R(\lambda)$.

Now, applying the Phragmén-Lindelöf theorem to the operator-function $R(\lambda)$ in each of the sectors (of angular measure 2ϵ) complementary to F_ϵ $(\epsilon < \pi/np)$, we obtain

$$|R(\lambda)| = O(|\lambda|^{2n-2}) \qquad (\lambda \to \infty) \tag{9.18}$$

over the entire complex plane. Since

$$R(\lambda) = (\widetilde{I} - \lambda\widetilde{A}_1)^{-1} = \widetilde{I} + \lambda\widetilde{A}_1 + \lambda^2\widetilde{A}_1^2 + \cdots,$$

we easily deduce from (9.18) that $\widetilde{A}_1^{2n-1} = 0$.

Bearing in mind that the operator $\widetilde{A}$ is representable in the form

$$\widetilde{A} = \widetilde{P}\widetilde{A}\widetilde{P} + \widetilde{P}\widetilde{A}\widetilde{Q} + \widetilde{Q}\widetilde{A}\widetilde{Q} = \widetilde{P}\widetilde{A}\widetilde{P} + \widetilde{P}\widetilde{A}\widetilde{Q} + A_1,$$

we obtain

$$\widetilde{A}^{2n-1} = (\widetilde{P}\widetilde{A}\widetilde{P})^{2n-1} + \widetilde{A}_1^{2n-1} + \widetilde{P}\widetilde{B}\widetilde{Q} = (\widetilde{P}\widetilde{A}\widetilde{P})^{2n-1} + \widetilde{P}\widetilde{B}\widetilde{Q},$$

where $\widetilde{B}$ is some completely continuous operator. From the last equality it follows that the range of the operator $\widetilde{A}^{2n-1}$ is contained in the subspace $\widetilde{P}\widetilde{\mathfrak{H}}$. But since the range of the operator $\widetilde{A}$, and consequently of the operator $\widetilde{A}^{2n-1}$, is dense in $\widetilde{\mathfrak{H}}$, the subspace $\widetilde{\mathfrak{H}}_0$ consists only of zero.

The theorem is proved for the bundle (9.6). The proof for the bundle (9.7) is analogous.

REMARK 9.1. It was shown in §8 that in the theorem of M. V. Keldyš concerning a linear bundle the condition $p(H) < \infty$ can be replaced by the condition $p([TH]_{\mathscr{I}}) < \infty$. This last condition is fulfilled, in particular, if $p(TH) < \infty$. Ju. A. Palant [1] has generalized Keldyš' theorem, showing that for a bundle of degree n the condition $p(H) < \infty$ can be replaced by the condition

$$p(H^r T_r) < \infty \quad (r = 1, 2, \cdots, n-1), \quad p(T_0 H^n) < \infty.$$

Another generalization of Theorem 9.1 to the case where the operator A is normal and satisfies certain additional conditions was obtained by D. È. Allahverdiev [1].

§10. Tests for the completeness of the systems of root vectors of unbounded operators

1. Let A be any closed linear operator having at least one regular point. A linear (in general, unbounded) operator B is said to be *A-completely continuous* if $\mathfrak{D}(A) \subseteq \mathfrak{D}(B)$ (i.e. the operator B is defined on the domain $\mathfrak{D}(A)$ of the operator A) and if for some regular point λ_0 of the operator A the operator $B(A - \lambda_0 I)^{-1}$ belongs to $\mathfrak{S}_\infty$.[27] One can show, in an obvious way, that in this case the operator $B(A - \lambda I)^{-1}$ belongs to $\mathfrak{S}_\infty$ for *every* regular point λ of the operator A. In particular, if the point $\lambda = 0$ is a regular point of the operator A, then the operator B will be A-completely continuous[28] if and only if

$$\mathfrak{D}(B) \supseteq \mathfrak{D}(A) \quad \text{and} \quad BA^{-1} \in \mathfrak{S}_\infty.$$

The definitions of a root vector, a normal eigenvalue and a normal point given in Chapter I for a bounded operator can be generalized in a natural way to closed operators.

LEMMA 10.1. *Let L be any linear selfadjoint operator, and T any L-completely continuous operator. Then the sets of normal points of the operators L and $A = L + T$ coincide. In particular, the entire nonreal spectrum of the operator A consists of normal eigenvalues. For any $\epsilon > 0$ there exists a number $r = r_\epsilon > 0$ such that the entire spectrum of the operator A will lie in the union of the disc $|\lambda| \leqq r$ and the two sectors*

$$(10.1) \qquad -\epsilon < \arg\lambda < \epsilon, \quad \pi - \epsilon < \arg\lambda < \pi + \epsilon.$$

Consequently if the point $\lambda = 0$ is a normal point of the operator L, then only a finite number of points of the spectrum of the operator A can lie outside the region (10.1).

PROOF. For any point λ from the domain of regularity $\rho(L)$ of the operator L we have

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[27] The definition of A-complete continuity can be given without any assumptions on the existence of regular points of A (see Gohberg and Kreĭn [1]).

[28] Hence the terminology "A-completely continuous" becomes natural.

$$A - \lambda I = (I + T(L - \lambda I)^{-1})(L - \lambda I). \tag{10.2}$$

The operator-function $T(L - \lambda I)^{-1}$, which can be represented in the form

$$T(L - \lambda I)^{-1} = T(L - iI)^{-1} - T(L - iI)^{-1}(L - \lambda I)^{-1}(i - \lambda),$$

is holomorphic in the region $\rho(L)$ and assumes values in $\mathfrak{S}_\infty$. Moreover, by Lemma 7.1 the limit relation

$$\lim_{\lambda \to \infty} |T(L - \lambda I)^{-1}| = 0$$

holds uniformly outside the sectors (10.1).

In particular, there exists a number $r = r_\varepsilon$ such that

$$|T(L - \lambda I)^{-1}| < 1$$

for points λ outside the sectors (10.1) with $|\lambda| > r$. For these λ the operator $A - \lambda I$ will obviously be invertible. Moreover, on the basis of Theorem I.5.1 it can be stated that the spectrum of A in $\rho(L)$ consists of isolated normal eigenvalues.

Finally, that the sets of normal points of the operators A and L coincide can be proved in the same way as the analogous assertion in Lemma I.5.2. This proof is based on the fact that

$$(A - \lambda I)^{-1} - (L - \lambda I)^{-1} = (A - \lambda I)^{-1}T(L - \lambda I)^{-1} \in \mathfrak{S}_\infty$$

for all $\lambda \in \rho(A) \cap \rho(L)$. The lemma is proved.

2. One says that a selfadjoint operator L has a discrete spectrum if its entire spectrum consists of normal eigenvalues, i.e. of isolated eigenvalues of finite multiplicity with a unique limit point at infinity. Obviously such an operator is always unbounded. If λ_0 is a regular point of the operator L, then the operator $(L - \lambda_0 I)^{-1}$ belongs to $\mathfrak{S}_\infty$.

THEOREM 10.1. *Let $A = L + T$, where L is a selfadjoint operator with a discrete spectrum, and T is an L-completely continuous operator such that*[29]

[29] Thus it is assumed in the condition (10.3) that the point $\lambda = 0$ is not an eigenvalue of the operator L. This restriction is not essential; if it is not fulfilled, then the condition (10.3) can be replaced by the condition $p[(L - aI)^{-1}T(L - aI)^{-1}] < \infty$, where a is any regular point of the operator L. It is easily seen that the last condition does not depend upon the choice of the real (or even complex) number a, regular for L, and corresponds to passing from the operator A to the operator $A - aI = (L - aI) + T$.

$$p(L^{-1}TL^{-1}) < \infty. \tag{10.3}$$

Then the entire spectrum of the operator A consists of normal eigenvalues. For any $\epsilon > 0$ all of them, with the possible exception of a finite number, lie in the sectors

$$-\epsilon < \arg\lambda < \epsilon, \quad \pi - \epsilon < \arg\lambda < \pi + \epsilon.$$

The system of root vectors of the operator A is complete in $\mathfrak{H}$.

PROOF. We may suppose, without loss of generality, that the point $\lambda = 0$ is not an eigenvalue of the operator A (cf. the footnote). Then, representing A in the form $A = (I + TL^{-1})\,L$, we find that $I + TL^{-1}$ vanishes only at zero. Bearing in mind that TL^{-1} is completely continuous, we conclude that $I + TL^{-1}$ is invertible and, moreover, that

$$(I + TL^{-1})^{-1} = I + S,$$

where S is some completely continuous operator.

Thus the operator A^{-1} admits the representation

$$A^{-1} = H(I + S),$$

where $H = L^{-1}$. From the relation

$$(I + TL^{-1})(I + S) = I$$

follows

$$S = -TL^{-1}(I + S).$$

Hence, by the condition (10.3), for some p $(< \infty)$ we have

$$HS = -L^{-1}TL^{-1}(I + S) \in \mathfrak{S}_p.$$

Thus by Theorem 8.1 and the additions to it which were made in §8.3, all the conclusions of Theorem 8.1 hold for the operator A^{-1}.

Since the eigenvalues of A^{-1} are the inverses of the eigenvalues of A, and the root vectors of A^{-1} and A coincide, the theorem is proved.

3. As a simple consequence of the theorem just proved, we obtain the following result.

The system of root vectors of the operator $A = L + K$ is complete whenever L is a selfadjoint operator with a discrete spectrum and K belongs to some $\mathfrak{S}_p$.

This result can be strengthened in a new direction, namely the condition $K \in \mathfrak{S}_p$ can be replaced by the condition $K \in \mathfrak{S}_\omega$. This follows from a general theorem of V. I. Macaev:

THEOREM 10.2 (MACAEV [2]). *Let $A = L + T$, where L is a selfadjoint operator with a discrete spectrum, and T is such that $\mathfrak{D}(T) \supset \mathfrak{D}(L)$ and $TL^{-1} \in \mathfrak{S}_\omega$. Then all the conclusions of Theorem* 10.1 *hold.*

This theorem can be obtained from the result of V. I. Macaev, indicated in §8.3, in the same way as Theorem 10.1 was obtained from Theorem 8.1.

After what has been said it is completely clear that from other theorems of the preceding sections one can obtain corresponding completeness theorems for specified classes of unbounded operators.

Other completeness theorems for unbounded operators are to be found in papers by M. A. Naĭmark [1], V. B. Lidskiĭ [5] and others. All of these theorems can be applied to various classes of differential operators.

We remark that on the basis of his Theorem 9.1, M. V. Keldyš [1] has obtained n-fold completeness for broad classes of ordinary and partial differential operator bundles.

§11. Asymptotic properties of the spectrum of a weakly perturbed positive operator

1. We shall present here certain of the results of M. V. Keldyš on asymptotic properties of operator bundles. We shall consider only linear operator bundles $I - T - \lambda H$, under the assumptions that H is a positive operator, $p(H) < \infty$, and the operator T belongs to $\mathfrak{S}_\infty$.[30)]

Under these assumptions (cf. §8.2), after an appropriate parameter shift ($\lambda \to \lambda + c$) the spectrum of the specified bundle can be regarded as the spectrum of an operator A of the form $A = H(I + S)$, where $S \in \mathfrak{S}_\infty$ and $(I + S)^{-1} \in \mathfrak{R}$.

For such operators the following lemma, which plays a decisive role in the sequel, is valid.

LEMMA 11.1. *Let $A = H(I + S)$, where $S \in \mathfrak{S}_\infty$, $(I + S)^{-1} \in \mathfrak{R}$, and H is a positive operator which belongs to the class $\mathfrak{S}_p$ for some integer $p > 0$. Then*

$$\lim_{\substack{\lambda \to \cdot \infty \\ \operatorname{Im}\lambda = 0}} \frac{\operatorname{sp}(A^p(\lambda))}{\operatorname{sp}(H^p(\lambda))} = 1,$$

30) M. V. Keldyš [1] published his results without proof. We have been able to find a proof of his theorem on the asymptotic behavior of the spectrum of a bundle for the case of a linear bundle.

where $A(\lambda)$ and $H(\lambda)$ are the Fredholm resolvents of A and H.

PROOF. We have

$$\begin{aligned} A(\lambda) - H(\lambda) &= (1/\lambda)\{(I-\lambda A)^{-1} - (I-\lambda H)^{-1}\} \\ &= -(I-\lambda A)^{-1} HS (I-\lambda H)^{-1} \\ &= -(I-\lambda A)^{-1} AT(I-\lambda H)^{-1} \\ &= -A(\lambda)\,T(I-\lambda H)^{-1}, \end{aligned}$$

where T denotes the operator $S(I+S)^{-1}$. Hence

$$A(\lambda) = H(\lambda)(I + C(\lambda)), \tag{11.1}$$

where

$$C(\lambda) = (I + T(I-\lambda H)^{-1})^{-1} - I.$$

According to Lemma 7.1, $|T(I-\lambda H)^{-1}| \to 0$ as $\lambda \to -\infty$, $\operatorname{Im}\lambda = 0$, and consequently

$$|C(\lambda)| \to 0 \quad \text{as} \quad \lambda \to -\infty, \operatorname{Im}\lambda = 0.$$

Raising both sides of (11.1) to the pth power, we obtain

$$\begin{aligned} A^p(\lambda) &= (H(\lambda) + H(\lambda)\,C(\lambda))^p \\ &= H^p(\lambda) + \sum H^{\alpha_1}(\lambda)\,C(\lambda)\,H^{\alpha_2}(\lambda)\,C(\lambda)\cdots H^{\alpha_r}(\lambda), \end{aligned} \tag{11.2}$$

where the summation is extended over all the $2^p - 1$ choices of integers $\alpha_1 > 0, \cdots, \alpha_{r-1} > 0$, $\alpha_r \geqq 0$ $(2 \leqq r \leqq p+1)$ such that $\alpha_1 + \cdots + \alpha_r = p$.

We choose N (>0) such that $|C(\lambda)| < 1$ for $\lambda < -N$. Then for $\lambda < -N$ we find, from (11.2):

$$\begin{aligned} |\operatorname{sp}(A^p(\lambda) - H^p(\lambda))| &\leqq |A^p(\lambda) - H^p(\lambda)|_1 \\ &\leqq \sum |H(\lambda)|_p^{\alpha_1} |C(\lambda)|\,|H(\lambda)|_p^{\alpha_2} |C(\lambda)| \cdots |H(\lambda)|_p^{\alpha_r} \\ &\leqq (2^p - 1)|H(\lambda)|_p^p |C(A)| = (2^p-1)\operatorname{sp}(H^p(\lambda))\,|C(\lambda)|. \end{aligned}$$

Hence

$$\left| \frac{\operatorname{sp}(A^p(\lambda))}{\operatorname{sp}(H^p(\lambda))} - 1 \right| = O(|C(\lambda)|) \quad \text{for } \lambda \to -\infty, \operatorname{Im}\lambda = 0,$$

and the lemma is proved.

LEMMA 11.2. *Under the conditions of Lemma 11.1 the limit relation*

$$\left\{ \int_0^\infty \frac{dn(r;A)}{(r+t)^p} \Big/ \int_0^\infty \frac{dn(r;H)}{(r+t)^p} \right\} \to 1 \quad \textit{as} \quad t \to +\infty \ (\operatorname{Im} t = 0)$$

holds, where $n(r;A)$ and $n(r;H)$ are the distribution functions of the characteristic numbers of the operators A and H.

PROOF. Since, together with $A \in \mathfrak{S}_p$, also $A(\lambda) \in \mathfrak{S}_p$ and hence $A^p(\lambda) \in \mathfrak{S}_1$, we have

$$\mathrm{sp}(A^p(\lambda)) = \sum_{j=1}^{\infty} \frac{1}{\mu_j(A^p(\lambda))} = \sum_{j=1}^{\infty} \frac{1}{(\mu_j(A) - \lambda)^p}, \tag{11.3}$$

and similarly

$$\mathrm{sp}(H^p(\lambda)) = \sum_{j=1}^{\infty} \frac{1}{(\mu_j(H) - \lambda)^p} \left(= \int_0^{\infty} \frac{dn(r;H)}{(r-\lambda)^p} \right). \tag{11.4}$$

Since

$$\int_0^{\infty} \frac{dn(r;A)}{(r+t)^p} = \sum_{j=1}^{\infty} \frac{1}{(|\mu_j(A)| + t)^p},$$

we conclude, taking (11.3), (11.4) and Lemma 11.1 into consideration, that to prove Lemma 11.2 it remains to derive the relation

$$\left\{ \sum_{j=1}^{\infty} \frac{1}{(\mu_j(A) + t)^p} \Big/ \sum_{j=1}^{\infty} \frac{1}{(|\mu_j(A)| + t)^p} \right\} \to 1 \quad \text{as} \quad t \to +\infty$$

$$(\mathrm{Im}\, t = 0).$$

According to Theorem 8.1, for any $\delta > 0$ almost all of the numbers $\mu_j(A)$ $(j = 1, 2, \cdots)$ lie inside the sector $|\arg \lambda| < \delta$. Since $\mu = \infty$ is their unique condensation point, this means that as $j \to \infty$

$$\mathrm{Re}\, \mu_j(A) \to \infty \quad \text{and} \quad \arg \mu_j(A) \to 0.$$

Hence for any $\epsilon > 0$ there exists $N = N_\epsilon > 0$ such that for $t > |A|$ the inequalities

$$\left| \left(\frac{\mu_j(A) + t}{|\mu_j(A)| + t} \right)^p - 1 \right| < \epsilon \qquad (j = N, N+1, \cdots)$$

will be fulfilled. Then for $t > |A|$ [31] we have

[31] If the inequalities $|b_j/a_j| < \epsilon$ $(j = 1, 2, \cdots, n)$ hold for the positive numbers a_j $(j = 1, 2, \cdots, n)$ and the complex numbers b_j $(j = 1, 2, \cdots, n)$, then

$$\left| \sum_{j=1}^{n} b_j \Big/ \sum_{j=1}^{n} a_j \right| < \epsilon.$$

$$\left|\left\{\sum_{j=N}^{\infty}\frac{1}{(\mu_j(A)+t)^p}\Big/\sum_{j=N}^{\infty}\frac{1}{(|\mu_j(A)|+t)^p}\right\}-1\right|\leqq\epsilon. \tag{11.5}$$

On the other hand, there is always a $T_\epsilon > 0$ such that for $t > T_\epsilon$

$$\left|\left\{\sum_{j=1}^{N-1}\frac{1}{(\mu_j(A)+t)^p}\Big/\sum_{j=1}^{N-1}\frac{1}{(|\mu_j(A)|+t)^p}\right\}-1\right|<\epsilon.$$

But then for $t > T_\epsilon$ the inequality (11.5) will hold with N replaced by 1. The lemma is proved.

2. To obtain the basic theorem of this section it remains to call upon a result of B. I. Korenbljum [**1**].[32)]

LEMMA 11.3. *Let* $\phi(r)$ and $\psi(r)$ $(0 \leqq r < \infty)$ *be nondecreasing functions for which, for some* p (> 0), *the integrals*

$$\begin{aligned}\Phi(t) &= \int_0^\infty \frac{d\phi(r)}{(r+t)^p},\\ \Psi(t) &= \int_0^\infty \frac{d\psi(r)}{(r+t)^p}\end{aligned}\qquad (0 \leqq t < \infty)$$

are finite, and

$$\lim_{t\to\infty}\frac{\Psi(t)}{\Phi(t)} = 1.$$

If $\phi(r) \to \infty$ *as* $r \to \infty$, *and there exists a positive constant* $\gamma < p$ *such that, for sufficiently large* $r < s$,

$$\frac{\phi(s)}{\phi(r)} \leqq \left(\frac{s}{r}\right)^\gamma, \tag{11.6}$$

then also

$$\lim_{r\to\infty}\frac{\psi(r)}{\phi(r)} = 1.$$

32) This result (and a still more general one) was obtained by B. I. Korenbljum [**1**], who generalized by his methods a Tauberian theorem of M. V. Keldyš [**1, 2**], in which more stringent requirements were imposed upon the function $\phi(x)$ and the number p, namely it was required that there exist a derivative $\phi'(x)$ such that $\alpha\phi(x) < x\phi'(x) < \beta\phi(x)$, where α and β are constants satisfying the conditions $0 < \beta < \alpha + 1$ and $p = [\beta] + 1$. Keldyš' theorem, in turn, was a generalization of well-known Tauberian theorems of Hardy and Littlewood (see Hardy [**1**]).

For the proof of Lemma 11.3 we refer the reader to the original paper by Korenbljum [1].

Combining Lemmas 11.2 and 11.3 leads to the basic theorem of M. V. Keldyš [1] in a somewhat refined form.[33)]

THEOREM 11.1. *Let* $A = H(I + S)$, *where* $S \in \mathfrak{S}_\infty$, $(I+S)^{-1} \in \mathfrak{R}$, *and* $H \in \mathfrak{S}_\infty$ *is a positive operator. If one can select a nondecreasing function* $\phi(r)$ $(0 \leqq r < \infty)$ *with the following properties*:

1) *for some* $\gamma > 0$, (11.6) *holds for sufficiently large* $r < s$,

$$\text{(11.7)} \qquad 2) \qquad \lim_{r\to\infty} (n(r;H)/\phi(r)) = 1,$$

then

$$\lim_{r\to\infty} \frac{n(r;A)}{n(r;H)} = 1, \tag{11.8}$$

where $n(r;H)$ *and* $n(r;A)$ *are the distribution functions of the characteristic numbers of the operators* H *and* A, *respectively.*

PROOF. We may assume without loss of generality that $\phi(r)$ is a positive function, with $\phi(0) = 0$ (otherwise, choosing $R > 0$ such that $\phi(r) > 0$ for $r > R$, we could then redefine the function $\phi(r)$ on $[0, R)$, setting $\phi(r) = 0$ for $0 \leqq r < R$).

Since, by virtue of (11.6), we have $\phi(r) = O(r^\gamma)$ $(r \to \infty)$, it follows that

$$\Phi(t) = \int_0^\infty \frac{d\phi(r)}{(r+t)^p}\,dt = p\int_0^\infty \frac{\phi(r)\,dr}{(r+t)^{p+1}} < \infty \qquad (0 < t < \infty).$$

Hence (11.7) implies that

$$F(t) = \int_0^\infty \frac{dn(r;H)}{(r+t)^p} = p\int_0^\infty \frac{n(r;H)}{(r+t)^{p+1}}\,dr < \infty$$

and

$$\lim_{t\to\infty} \frac{F(t)}{\Phi(t)} = 1.$$

Thus $H \in \mathfrak{S}_p$ and according to Lemma 11.2

33) This refinement arose automatically in connection with the replacement of Keldyš' Tauberian theorem by the Tauberian theorem of Korenbljum [1].

$$\lim_{t\to\infty} \frac{G(t)}{F(t)} = 1,$$

where

$$G(t) = \int_0^\infty \frac{dn(r; A)}{(r+t)^p}.$$

Consequently $G(t)/\Phi(t) \to 1$ as $t \to \infty$, and the relation (11.8) is obtained on the basis of Lemma 11.3. The theorem is proved.

We remark that for a nondecreasing differentiable function $\phi(r)$ satisfying the condition

$$\beta = \overline{\lim_{r\to\infty}}\, r \frac{\phi'(r)}{\phi(r)} < \infty, \tag{11.9}$$

condition 1) of the theorem will always be fulfilled for any $\gamma > \beta$. In particular, the conditions (11.9) and (11.6) will be satisfied by every nondecreasing function $\phi(r) = r^{1/p}L(r)$, where $p > 0$ and $L(r)$ is a slowly varying function.

REMARK 11.1. Theorem 11.1 remains valid if in its statement the condition that the operator $H \in \mathfrak{S}_\infty$ be positive is replaced by the following two conditions:

a) H is a complete selfadjoint operator,

b) H has at most a finite number of negative eigenvalues.

In this extended version Theorem 11.1 is equivalent to the following result.

Let $H \in \mathfrak{S}_\infty$ be an operator having properties a), b), *and let $T \in \mathfrak{S}_\infty$. If the distribution function $n(r; H)$ fulfills the conditions of Theorem* 11.1, *then the limit relation*

$$\lim_{r\to\infty} \frac{n(r; L)}{n(r; H)} = 1 \tag{11.10}$$

holds for the distribution function $n(r; L)$ of the the characteristic numbers of the bundle $L(\lambda) = I - T - \lambda H$.

This result, roughly speaking, means that in seeking the asymptotic behavior of the characteristic numbers of the bundle $L(\lambda)$ one can discard the completely continuous term T. We shall show that, following certain transformations of the bundle $L(\lambda)$, this result can be obtained as a corollary of Theorem 11.1.

In fact, for the Fredholm resolvent $H(-t) = H(I + tH)$ [1] we have

$\mu_j(H(-t)) = \mu_j(H) + t$. Therefore one can choose t positive and sufficiently large so that all the characteristic numbers of the resolvent $H(-t)$ will be positive. Then the characteristic numbers of the bundle

$$L_1(\lambda) = I - T(-t) - \lambda H(-t) (= (I + tH)^{-1} L(\lambda - t)),$$

where $T(-t) = (I + tH)^{-1} T$, can be obtained from the characteristic numbers of the bundle $L(\lambda)$ by the shift $\lambda \to \lambda + t$.

On the other hand, choosing t so that the operator $T(-t)$ has norm less than 1, one can assert that the characteristic numbers of the bundle $L_1(\lambda)$ coincide with the characteristic numbers of the operator

$$A_1 = H(-t)(I + S(-t)),$$

where $S(-t) = (I - T(-t))^{-1} - I \ (\in \mathfrak{S}_\infty)$.

It follows from the relation (11.6) that $\phi(r-t)/\phi(r) \to 1$ as $r \to \infty$, and consequently the relation

$$\lim_{r \to \infty} \frac{n(r; H_1)}{\phi(r)} = 1$$

is valid for the distribution function $n(r; H_1) = n(r - t; H)$. Thus Theorem 11.1 is applicable to the operator A_1, and so

(11.11) $$\lim_{r \to \infty} \frac{n(r; A_1)}{n(r; H_1)} = 1.$$

Bearing in mind, finally, that $n(r; A_1) = n(r - a; L)$ $(r \geqq a)$, we obtain the relation (11.10) from (11.11).

Theorem 11.1 also remains valid if the operator $A = H(I + S)$ which appears in it is replaced by the operator $A_1 = (I + S_1) H (I + S_2)$, where $S_j \in \mathfrak{S}_\infty$ and $(I + S_j)^{-1} \in \mathfrak{R}$ $(j = 1, 2)$. Indeed, such an operator, as was mentioned in Remark 8.1, is similar to the operator $A_1 = H(I + S_2) \cdot (I + S_1)$ $(A_1 = (I + S_1)^{-1} A (I + S_1))$, to which Theorem 11.1 is applicable. This remark makes it possible to derive from Theorem 11.1 an asymptotic relation for the s-numbers of the operator $A = H(I + S)$. However, one can independently prove a stronger assertion for the s-numbers of an operator of this form than that which is obtained on the basis of Theorem 11.1.

THEOREM 11.2. *Let $H \in \mathfrak{S}_\infty$ be an infinite-dimensional selfadjoint operator, let $S \in \mathfrak{S}_\infty$, and suppose moreover that at least one of the following two conditions is fulfilled:*

(11.12) 1) *H is a complete operator and $(I + S)^{-1} \in \mathfrak{R}$;*

(11.13) 2) $\lim_{n\to\infty} |s_{n+1}(H)/s_n(H)| = 1.$

Then for the operator $A = H(I+S)$ *one has the limit relation*

$$\lim_{n\to\infty} \frac{s_n(A)}{s_n(H)} = 1. \tag{11.14}$$

It is not difficult to see that this theorem follows from Theorem 11.3.

THEOREM 11.3 (M. G. KREĬN). *Suppose that the selfadjoint operators* $H = H^* \in \mathfrak{S}_\infty$ *and* $S \in \mathfrak{S}_\infty$ *satisfy at least one of the conditions* 1), 2) *of Theorem* 11.2. *Then for the operator* $A = H(I+S)H$ *one has the limit relation*

$$\lim_{n\to\infty} \frac{\lambda_n(A)}{\lambda_n(H^2)} = 1. \tag{11.15}$$

In fact, if the operator $A = H(I+S)$ satisfies the conditions of Theorem 11.2, then Theorem 11.3 is applicable to the operator $A_1 = AA^* = H(I+S_1)H$, where $S_1 = S + S^* + SS^*$ $(\in \mathfrak{S}_\infty)$, and the relation (11.14) is obtained as a consequence of the relation (11.15) for the operator A_1.

PROOF. Let $\{\phi_j\}_1^\infty$ be a complete orthonormal system of eigenvectors of the operator H:

$$H\phi_j = s_j(H)\phi_j \qquad (j = 1, 2, \cdots).$$

We denote by $\mathfrak{L}_n$ $(n = 1, 2, \cdots)$ the linear hull of the vectors $\phi_1, \phi_2, \cdots, \phi_n$. Then

$$\lambda_n(H^2) = \max_{\phi \in \mathfrak{L}_{n-1}^\perp} \frac{(H\phi, H\phi)}{(\phi, \phi)} \qquad (n = 1, 2, \cdots; \ \mathfrak{L}_0 = 0). \tag{11.16}$$

For all vectors ϕ $(\neq 0)$ of the subspace $\mathfrak{L} = \overline{\mathfrak{R}(H)}$ we have

$$\frac{(A\phi, \phi)}{(\phi, \phi)} = \frac{(H\phi, H\phi)}{(\phi, \phi)} \left(1 + \frac{(SH\phi, H\phi)}{(H\phi, H\phi)}\right). \tag{11.17}$$

Since if the vector ϕ belongs to the subspace $\mathfrak{N}_{n-1} = \mathfrak{L} \ominus \mathfrak{L}_{n-1}$, the vector $\psi = H\phi$ belongs to the same subspace, we conclude that the quantity

$$\epsilon_n = \sup_{\phi \in \mathfrak{N}_{n-1}} \left| \frac{(SH\phi, H\phi)}{(H\phi, H\phi)} \right| = \sup_{\psi \in \mathfrak{N}_{n-1}} \frac{|(S\psi, \psi)|}{(\psi, \psi)}$$

tends to 0 as $n \to \infty$.

Thus it follows from (11.17) that for sufficiently large n

$$(A\phi, \phi) \geqq 0 \qquad (\phi \in \mathfrak{N}_{n-1}).$$

Taking into consideration, further, that $A\phi = 0$ for all $\phi \in \mathfrak{L}^{\perp}$, we conclude that the operator A has at most a finite number of negative eigenvalues.

We shall show that without loss of generality the operator A can be assumed nonnegative. To do this, we consider the Fredholm resolvent

$$A(-t) = A(I + tA)^{-1}$$

of the operator A. The eigenvalues $\lambda(A(-t))$ and $\lambda(A)$ are related by

$$\lambda^{-1}(A(-t)) = \lambda^{-1}(A) + t, \tag{11.18}$$

and thus for sufficiently large t the operator $A(-t)$ is nonnegative. On the other hand, inserting the expression for A into the equality $A(-t) = A - tA(-t)A$, we obtain $A(-t) = TH$, where $T = (I - tA(-t))H(I + S) \in \mathfrak{S}_{\infty}$. But then

$$A(-t) = A - tAA(-t) = H(I + S(-t))\,H,$$

where $S(t) = S - t(I + S)\,H(I - tA(-t))\,H(I + S) \quad (\in \mathfrak{S}_{\infty})$. The operator-function $S(t)$ depends analytically on the parameter t, and consequently if the operator $I + S$ $(= I + S(0))$ is invertible, then by Theorem I.5.1 the number t (> 0) can be chosen so that at the same time the operator $I + S(t)$ will be invertible. Finally, we note that by virtue of the relation (11.18) the limit relation (11.15) for the operator A is equivalent to the same relation for the operator $A(-t)$.

Thus we can assume that the operator A is nonnegative. Then

$$\lambda_n(A) \leqq \sup_{\phi \in \mathfrak{N}_{n-1}} \frac{(A\phi, \phi)}{(\phi, \phi)}$$

$$= \sup_{\phi \in \mathfrak{N}_{n-1}} \left[\frac{(H\phi, H\phi)}{(\phi, \phi)} \left(1 + \frac{(SH\phi, H\phi)}{(H\phi, H\phi)} \right) \right].$$

Hence, by (11.16) we obtain the inequality

$$\lambda_n(A) \leqq \lambda_n(H^2)(1 + \epsilon_n),$$

and consequently

$$\overline{\lim_{n \to \infty}} \frac{\lambda_n(A)}{\lambda_n(H^2)} \leqq 1. \tag{11.19}$$

We note that so far in our proof neither of the conditions 1) and 2) of Theorem 11.2 has been used.

Let us now assume that the property 1) is fulfilled, i.e. H is a complete operator and the operator $I+S$ is invertible. It follows from the equality

$$(A\phi,\phi) = ((I+S)\psi,\psi), \qquad \psi = H\phi,$$

that the operator $I+S$ is nonnegative.

Let us consider the operator

$$F = (I+S)^{1/2}H.$$

Since $F^*F = A$, one has for the operator

$$A_1 = FF^* = (I+S)^{1/2}H^2(I+S)^{1/2}$$

the relations

$$\lambda_n(A_1) = s_n^2(F) = \lambda_n(A) \qquad (n=1,2,\cdots). \tag{11.20}$$

Since the operator $I+S$ is invertible, $(I+S)^{-1} = I+S_1$ $(S_1 \in \mathfrak{S}_\infty)$ and $(I+S_1)^{1/2} = (I+S)^{-1/2}$. Consequently

$$H^2 = (I+S_1)^{1/2}A_1(I+S_1)^{1/2}.$$

Setting $L = A_1^{1/2}(I+S_1)^{1/2}$, we will have $H^2 = L^*L$. But then, for $H_1 = LL^*$ we will have

$$H_1 = A_1^{1/2}(I+S_1)A_1^{1/2}, \quad \lambda_n(H_1) = \lambda_n(H^2) \qquad (n=1,2,\cdots). \tag{11.21}$$

Putting $A = H_1$ and $H^2 = A_1$ in the relation (11.19) which has already been proved, we obtain

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n(H_1)}{\lambda_n(A_1)} \leqq 1.$$

Thus by (11.20) and (11.21)

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n(H^2)}{\lambda_n(A)} \leqq 1. \tag{11.22}$$

The relation (11.22) together with (11.19) yields (11.15).

We now consider the case in which the operator $I+S$ is not invertible. In this case we may assume without loss of generality that the operator H is complete. In fact, the subspace $\overline{\mathfrak{R}(H)}$ $(\supseteq \overline{\mathfrak{R}(A)})$ is in this case a common invariant subspace for A and H, and A and H are inessential extensions of their restrictions $\hat{A}$ and $\hat{H}$ to $\overline{\mathfrak{R}(H)}$.

Since $\hat{A} = \hat{H}(I + PSP)\hat{H}$, where P is the orthoprojector which projects $\mathfrak{H}$ onto $\overline{\mathfrak{R}(H)}$, we may assume without loss of generality that $\overline{\mathfrak{R}(H)} = \mathfrak{H}$.

Let us denote by $\mathfrak{Z}$ the zero set of the operator $I + S$. It will be the zero set of the operator A_1, so that $\mathfrak{H}_1 = \overline{\mathfrak{R}(A_1)} = \mathfrak{Z}^{\perp}$. Let Q be the the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{H}_1$; then

$$A_1 = QA_1Q = (I + S)^{1/2}QH^2Q(I + S)^{1/2},$$

which can be written as

$$\hat{A}_1 = (I + \hat{S})^{1/2}\hat{H}(I + \hat{S})^{1/2},$$

where $\hat{A}_1$, $\hat{S}$, $\hat{H}$ are the restrictions of the operators A_1, S and QH^2Q to $\mathfrak{H}_1$.

All the conditions of the theorem, including the invertibility of $I + \hat{S}$, will be fulfilled by the operators $\hat{A}$ and $\hat{H}$, which are related by

$$\hat{A} = \hat{H}^{1/2}(I + \hat{S})\hat{H}^{1/2}.$$

Therefore

$$\lim_{n\to\infty} \frac{\lambda_n(\hat{A})}{\lambda_n(\hat{H})} = 1.$$

On the other hand,

$$\lambda_n(\hat{A}) = \lambda_n(A),$$

and by virtue of Corollary II.2.1

$$\lambda_{n+\nu}(H^2) \leqq \lambda_n(\hat{H}) \leqq \lambda_n(H^2),$$

where ν is the dimension of $\mathfrak{Z}$. Thus we will have

$$\varliminf_{n\to\infty} \frac{\lambda_n(A)}{\lambda_{n+\nu}(H^2)} \geqq 1.$$

By condition 2), it follows that

$$\varliminf_{n\to\infty} \frac{\lambda_n(A)}{\lambda_n(H^2)} \geqq 1.$$

The last relation, together with (11.19), yields (11.15). The theorem is proved.

REMARK 11.2. Theorem 11.3 admits the following generalization. *Let $G \in \mathfrak{S}_\infty$ be an infinite-dimensional operator, and $S \in \mathfrak{S}_\infty$ a self-adjoint operator, and suppose moreover that at least one of the following*

two conditions is fulfilled:

1) $\mathfrak{N}(G)$ *is dense in* $\mathfrak{H}$ *and* $(I+S)^{-1} \in \mathfrak{R}$.

2) $$\lim_{n\to\infty} \frac{s_{n+1}(G)}{s_n(G)} = 1.$$

Then the relation

$$\lim_{n\to\infty} \frac{\lambda_n(A)}{\lambda_n(G^*G)} = 1$$

holds for the operator $A = G^*(I+S)G$.

Naturally, for the case of a selfadjoint operator A Theorem 11.1 admits the following strengthening:

THEOREM 11.4 (M. G. KREĬN [**10**]).[34] *Let* H $(\in \mathfrak{S}_\infty)$ *be an infinite-dimensional nonnegative operator, and* A *a selfadjoint operator of the form*

$$A = H(I+S),$$

where $S \in \mathfrak{S}_\infty$, *and suppose moreover that at least one of the following two conditions is fulfilled:*

1) $$(I+S)^{-1} \in \mathfrak{R},$$

2) $$\lim_{n\to\infty} (\lambda_{n+1}(H)/\lambda_n(H)) = 1.$$

Then

$$\lim_{n\to\infty} \frac{\lambda_n(A)}{\lambda_n(H)} = 1. \tag{11.23}$$

PROOF. Let us first consider the case 1), in which $I+S$ is invertible. Without loss of generality we may suppose that the operator H is complete. In fact, from the equality $A = H(I+S) = (I+S^*)H$ it follows that the subspace $\overline{\mathfrak{N}(H)}$ is invariant with respect to the operators A and $I+S^*$, and $A[\mathfrak{N}(H)^\perp] = 0$. Let us denote by $\hat{A}$, $\hat{H}$ and S_1^* the restrictions of A, H and S^* to the subspace $\overline{\mathfrak{N}(H)}$. The operaror $\hat{A}$ can obviously be represented in the form $\hat{A} = (I+S_1^*)\hat{H}$. Since $\hat{A}$ and $\hat{H}$ are selfadjoint, we obtain $\hat{A} = \hat{H}(I+S_1)$. Obviously $S_1 = PSP$, where P is the orthoprojector which projects $\mathfrak{H}$ onto $\overline{\mathfrak{N}(H)}$.

[34] By an oversight the requirement that one of the two conditions 1), 2) be fulfilled was omitted in [**10**].

From the invertibility of the operator $I + S^*$ in $\mathfrak{H}$ follows the invertibility of the operator $I + S_1^*$, and also that of the operator $I + S_1$, in $\overline{\mathfrak{R}(H)}$. Thus if H were not complete we could pass to the consideration of the restriction $\hat{A}$ on the subspace $\overline{\mathfrak{R}(H)}$.

By Theorem 11.2 the relation (11.14) holds for the operator A. This relation is equivalent to the relation (11.23), since by Lemma 7.1 all the eigenvalues of A, with the possible exception of a finite number, are positive, and consequently, starting with some n, $\lambda_n(A) = s_n(A)$.

Let us now consider the case in which condition 2) is fulfilled, and the operator $I + S$ is not invertible. In this case $\mathfrak{H} = \mathfrak{L} \dot{+} \mathfrak{H}_1$, where $\mathfrak{L}$ is the root subspace of the operator S corresponding to the eigenvalue -1, and $\mathfrak{H}_1$ is an invariant subspace of S in which $I + S$ is invertible. Let Q be the orthoprojector which projects $\mathfrak{H}$ onto $\mathfrak{H}_1$. Its complement $P = I - Q$ will have finite dimension ν, equal to the dimension of $\mathfrak{L}$.

Since $(I + S)\,Q = Q(I + S)\,Q$, we have

$$QAQ = QHQQ(I + S)\,Q. \tag{11.24}$$

Let us denote by A_1, H_1 and S_1 the operators induced in $\mathfrak{H}_1$ by the operators QAQ, QHQ and S, respectively. The relation (11.24) means that

$$A_1 = H_1(I + S_1).$$

Since H is positive, the operator H_1 is positive. Since moreover $I + S_1$ is invertible, we have, as was already proved,

$$\lim_{n\to\infty} \frac{\lambda_n(A_1)}{\lambda_n(H_1)} = 1. \tag{11.25}$$

Obviously $\lambda_n(H_1) = \lambda_n(QHQ)$ $(n = 1, 2, \cdots)$, and the operator

$$QHQ = (I - P)\,H(I - P) = H - P(H - HP) - HP$$

differs from H by an operator of finite dimension $\leqq 2\nu$.

Therefore (see Corollary II.2.1)

$$\lambda_{n+2\nu}(H) \leqq \lambda_n(H_1) \leqq \lambda_n(H), \tag{11.26}$$

and similarly, we may without loss of generality assume that the operator A_1 is nonnegative:

$$s_{n+2\nu}(A) \leqq s_n(A_1) \leqq s_n(A). \tag{11.27}$$

Comparing (11.23), (11.24) and (11.25), we see that

(11.28)
$$\lim_{n\to\infty} \frac{s_n(A)}{\lambda_{n+2\nu}(H)} \geqq 1, \quad \overline{\lim_{n\to\infty}} \frac{s_n(A)}{\lambda_{n-2\nu}(H)} \leqq 1,$$

and since, according to (11.11),

$$\lim_{n\to\infty} \frac{\lambda_n(H)}{\lambda_{n+2\nu}(H)} = \lim_{n\to\infty} \frac{\lambda_{n-2\nu}(H)}{\lambda_n(H)} = 1,$$

(11.23) follows from (11.28).

It remains now to take into consideration that A has at most a finite number of negative eigenvalues, i.e. for sufficiently large n we have $\lambda_n(A) = s_n(A)$. This property of A follows from the fact that the operator $A - A_1$ is finite-dimensional, and, as was proved, A_1 has at most a finite number of negative eigenvalues. The theorem is proved.

It is not difficult to see that if, corresponding to the operator H, there exists a function $\phi(r)$ $(0 \leqq r < \infty)$ which has properties 1) and 2) of Theorem 11.1, then condition 2) of Theorem 11.4 will always be fulfilled, and the relations (11.23) and (11.8) will be equivalent.

§12. Selfadjoint quadratic bundles[35)]

1. In the linear theory of small damped oscillations of systems with an infinite number of degrees of freedom, an important role is played by the properties of quadratic bundles $L(\lambda)$ of the form

$$L(\lambda) = I + \lambda B + \lambda^2 C,$$

where C is a positive completely continuous operator and B is a nonnegative bounded operator.

Everywhere henceforth, if no mention is made to the contrary, we shall assume that the first condition and also the condition

(12.1)
$$B = B^* \in \Re$$

hold.

Since the equation $L(\lambda)\,\phi = 0$ is equivalent to the pair of equations

$$\begin{cases} \phi = -\lambda(B\phi + C\psi), \\ \psi = \lambda\phi, \end{cases}$$

the spectrum of characteristic numbers of the bundle $L(\lambda)$ coincides

35) The content of this section is taken from the work of M. G. Kreĭn and H. K. Langer [1] and [2] (§§2, 7).

with the spectrum of characteristic numbers of the operator $\mathscr{A}_1$, acting in the space $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$ (the orthogonal sum of two copies of the space $\mathfrak{H}$) and defined by

$$\mathscr{A}_1 = \begin{pmatrix} -B & -C \\ I & 0 \end{pmatrix}.$$

Moreover, on the basis of the general considerations of §9 one can also assert that the algebraic and geometric multiplicities of any characteristic number of the operator $\mathscr{A}_1$ and the bundle $L(\lambda)$ are the same, and that the two-fold completeness in $\mathfrak{H}$ of the system of eigenvectors and associated vectors of the bundle L is equivalent to the completeness of the system of root vectors of the operator $\mathscr{A}_1$ in $\tilde{\mathfrak{H}}$.

Instead of the operator $\mathscr{A}_1$, it will be convenient for us to deal with the operator

$$\mathscr{A} = \begin{pmatrix} -B & -C^{1/2} \\ C^{1/2} & 0 \end{pmatrix},$$

which acts in the same space $\tilde{\mathfrak{H}}$. Further introducing the operators

$$\mathscr{S} = \begin{pmatrix} I & 0 \\ 0 & C^{1/2} \end{pmatrix}, \quad \mathscr{G} = \begin{pmatrix} -B & -C^{1/2} \\ I & 0 \end{pmatrix},$$

we will have

(12.2) $$\mathscr{A}_1 = \mathscr{G}\mathscr{S}, \quad \mathscr{A} = \mathscr{S}\mathscr{G},$$

and consequently

(12.3) $$\mathscr{S}\mathscr{A}_1 = \mathscr{A}\mathscr{S}, \quad \mathscr{A}_1\mathscr{G} = \mathscr{G}\mathscr{A}.$$

It is easily seen that the operators $\mathscr{A}_1$, $\mathscr{A}$, $\mathscr{S}$, $\mathscr{G}$ and their adjoints vanish only at zero.

If $\{x_j\}_0^l$ $(l \geqq 0)$ is some Jordan chain for the operator $\mathscr{A}_1$, corresponding to the eigenvalue λ, i.e.

(12.4) $$\mathscr{A}_1 x_0 = \lambda x_0, \quad \mathscr{A}_1 x_j = \lambda x_j + x_{j-1} \qquad (j = 1, 2, \cdots, l),$$

then the vectors $y_j = \mathscr{S} x_j$ $(j = 0, 1, \cdots, l)$ constitute a Jordan chain for the operator $\mathscr{A}$, corresponding to the same eigenvalue λ, i.e.,

(12.5) $$\mathscr{A} y_0 = \lambda y_0, \quad \mathscr{A} y_j = \lambda y_j + y_{j-1} \qquad (j = 1, 2, \cdots, l).$$

In fact, the last relations are obtained from the relations (12.4) by termwise application of the operator $\mathscr{S}$ (taking into account the first of the equalities (12.3)).

Conversely, suppose that the relations (12.5) hold. We put

$$x_0 = \lambda^{-1}\mathscr{G} y_0, \quad x_j = \lambda^{-1}(\mathscr{G} y_j - x_{j-1}) \qquad (j = 1, \cdots, l).$$

By virtue of the first of the equalities (12.2) we obtain $y_j = \mathscr{S} x_j$ $(j = 0, 1, \cdots, l)$. Therefore, using the first of the equalities (12.3), we can rewrite the relations (12.5) in the form

$$\mathscr{S}(\mathscr{A}_1 x_0 - \lambda x_0) = 0, \quad \mathscr{S}(\mathscr{A}_1 x_j - \lambda x_j - x_{j-1}) = 0 \qquad (j = 1, 2, \cdots, l).$$

Since the operator $\mathscr{S}$ vanishes only at zero, the relations (12.4) follow.

We have thus proved that the operator $\mathscr{S}$ maps every root subspace $\mathfrak{L}_\lambda(\mathscr{A}_1)$ of the operator $\mathscr{A}_1$ onto the root subspace $\mathfrak{L}_\lambda(\mathscr{A})$ of the operator $\mathscr{A}$ in a one-to-one fashion: $\mathscr{S}\mathfrak{L}_\lambda(\mathscr{A}_1) = \mathfrak{L}_\lambda(\mathscr{A})$. At the same time, we have proved that $\mathscr{G}\mathfrak{L}_\lambda(\mathscr{A}) = \mathfrak{L}_\lambda(\mathscr{A}_1)$. Since $\overline{\Re(\mathscr{S})} = \tilde{\mathfrak{H}}$ and $\overline{\Re(\mathscr{G})} = \tilde{\mathfrak{H}}$, it follows that the completeness of the system of root vectors of one of the operators $\mathscr{A}$ and $\mathscr{A}_1$ implies the same completeness for the other operator.

It follows that the system of eigenvectors and associated vectors of the bundle $L(\lambda)$ is two-fold complete if and only if the system of root vectors of the operator $\mathscr{A}$ is complete.

Incidentally, in deriving this assertion it was nowhere used that $B = B^*$.

2. Under the assumption (12.1) we will have

$$\mathscr{A}_{\mathscr{R}} = \begin{pmatrix} -B & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{A}_{\mathscr{I}} = \frac{1}{i}\begin{pmatrix} 0 & -C^{1/2} \\ C^{1/2} & 0 \end{pmatrix}.$$

It is not difficult to see that the spectrum of the selfadjoint operator $\mathscr{A}_{\mathscr{I}}$ is situated symmetrically about zero and moreover that

$$\lambda_n^{\pm}(\mathscr{A}_{\mathscr{I}}) = \pm\lambda_n(C^{1/2}) = \pm\lambda_n^{1/2}(C) \qquad (n = 1, 2, \cdots). \tag{12.6}$$

We note that the operators $\mathscr{A}$ and $\mathscr{A}^*$ are similar:

$$\mathscr{A}^* = \mathscr{Y}^{-1}\mathscr{A}\mathscr{Y}, \quad \text{where} \quad \mathscr{Y} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{12.7}$$

Since our further considerations will be partly concerned with the case $B \notin \mathfrak{S}_\infty$, we shall say, for clarity, that a point $\lambda = \lambda_0$ is a *regular*

point of the bundle L if the operator $L(\lambda_0)$ is invertible. We denote the set of regular points of the bundle L by $\rho(L)$. Its complement is called the *spectrum* of the bundle L and is denoted by $\sigma(L)$.

I. *The spectrum $\sigma(L)$ is symmetric with respect to the real axis, i.e., $\sigma(L) = \overline{\sigma(L)}$. Every nonreal point $\lambda_0 \in \sigma(L)$ is an isolated characteristic number of the bundle $L(\lambda)$ of finite algebraic multiplicity.*[36] *The root lineals $\mathfrak{L}_{\lambda_0}(L)$ and $\mathfrak{L}_{\bar\lambda_0}(L)$ have the same dimension and, moreover, the same structure.*

The first assertion follows from the equality $L(\bar\lambda) = [L(\lambda)]^*$, which follows from the conditions $B = B^*$, $C = C^*$.

It is obvious that a sufficiently small neighborhood of zero consists of regular points of the bundle L. For nonreal λ, the operator $I + \lambda B$ is invertible, and for such λ

$$L(\lambda) = (I + \lambda B)(I + T(\lambda)),$$

where $T(\lambda) = \lambda^2(I + \lambda B)^{-1}C$ is a holomorphic function with values in $\mathfrak{S}_\infty$. Therefore on the basis of Theorem I.5.2 we can assert that the nonreal part of the spectrum $\sigma(L)$ consists of isolated characteristic numbers.

If λ_0 is a nonreal characteristic number of the bundle L, then it will be a characteristic number of the operator $\mathscr{A}_1$ and consequently of the operator $\mathscr{A}$. Since $\mathscr{A}_{\mathscr{I}} \in \mathfrak{S}_\infty$, λ_0 will be a normal eigenvalue of $\mathscr{A}$ and the root lineal $\mathfrak{L}_\lambda(\mathscr{A})$ will be finite-dimensional. Hence

$$\dim \mathfrak{L}_{\lambda_0}(L) = \dim \mathfrak{L}_{\lambda_0}(\mathscr{A}_1) = \dim \mathfrak{L}_{\lambda_0}(\mathscr{A}) < \infty.$$

To prove the third assertion in I we remark that by virtue of (12.7) the root lineals $\mathfrak{L}_{\lambda_0}(\mathscr{A})$ and $\mathfrak{L}_{\bar\lambda_0}(\mathscr{A}) = \mathscr{Y}\mathfrak{L}_{\bar\lambda_0}(\mathscr{A}^*)$ $(\lambda_0 \neq \bar\lambda_0)$ have the same dimension and the same structure (the operator $\mathscr{Y}^{-1}$ maps every Jordan chain for $\mathscr{A}$ from $\mathfrak{L}_{\bar\lambda_0}(\mathscr{A})$ into a Jordan chain for $\mathscr{A}^*$ from $\mathfrak{L}_{\bar\lambda_0}(\mathscr{A}^*)$); therefore the root lineals

$$\mathfrak{L}_{\lambda_0}(\mathscr{A}_1) = \mathscr{S}^{-1}\mathfrak{L}_{\lambda_0}(\mathscr{A}) \quad \text{and} \quad \mathfrak{L}_{\bar\lambda_0}(\mathscr{A}_1) = \mathscr{S}^{-1}\mathfrak{L}_{\bar\lambda_0}(\mathscr{A})$$

have the same dimension and the same structure with respect to the operator $\mathscr{A}_1$, from which the third assertion follows.

II. *If $B \geq 0$, then the spectrum $\sigma(L)$ lies in the left halfplane* $\operatorname{Re}\lambda \leq 0$.

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[36] Every real point $\lambda_0 \in \sigma(L)$ such that $-\lambda_0^{-1}$ does not belong to the condensed spectrum (see Chapter II, §7) of B has this property. We mention in passing that every point λ_0 for which $-\lambda_0^{-1}$ belongs to the condensed spectrum of B lies in the spectrum $\sigma(L)$.

In fact, if $\alpha = \operatorname{Re}\lambda > 0$, the relation

$$[\lambda^{-1}L(\lambda)]_{\mathscr{R}} = \frac{\alpha}{|\lambda|^2} I + B + \alpha C \geqq \frac{\alpha}{|\lambda|^2} I$$

is fulfilled; it follows that the operator $L(\lambda)$ is invertible.[37]

3. If $B \in \mathfrak{S}_\infty$, the spectrum $\sigma(L)$ consists only of isolated characteristic numbers of finite algebraic multiplicity with a unique limit point at infinity, if there are infinitely many of them. In this case the spectrum $\sigma(L)$ can turn out to be empty. Indeed, let $B = Z + Z^*$, $C = Z^*Z$, where Z is a Volterra operator. Then for any λ the operator

$$L(\lambda) = (I + \lambda Z^*)(I + \lambda Z)$$

is invertible.

If $B = B^* \notin \mathfrak{S}_\infty$, then the spectrum $\sigma(L)$ always contains some real point.

But it turns out that for any operator $B = B^* \in \mathfrak{R}$ the following result is valid.

THEOREM 12.1. *The set of nonreal characteristic numbers of the bundle L can have at most one limit point, which is infinity. If $\{\check{\lambda}_j(L)\}_1^\omega$ $(\omega \leqq \infty)$ is the complete sequence of characteristic numbers of the bundle L, arranged in order of increasing modulus $(|\check{\lambda}_1(L)| \leqq |\check{\lambda}_2(L)| \leqq \cdots)$, then for any function $f(r)$ $(0 < r < \infty;\ f(+0) = 0)$ for which the function $f(e^t)$ $(-\infty < t < \infty)$ is convex one has the inequalities*

$$\sum_{j=1}^{n} f\left(\frac{1}{|\check{\lambda}_j(L)|}\right) \leqq \sum_{j=1}^{n} f(\sqrt{\lambda_j(C)}) \qquad (n = 1, 2, \cdots, \omega). \tag{12.8}$$

In particular, if $\omega = \infty$, then

$$\sum_{j=1}^{\infty} f\left(\frac{1}{|\check{\lambda}_j(L)|}\right) \leqq \sum_{j=1}^{\infty} f(\sqrt{\lambda_j(C)}) \qquad (= \operatorname{sp} f(C^{1/2})). \tag{12.9}$$

If it is assumed that the function $f(e^t)$ is strictly convex and the right side of (12.9) is finite, the the equal sign will hold if and only if the operators B and C commute and $B^2 < 4C$.

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[37] In fact, if $A = A_{\mathscr{R}} + iA_{\mathscr{J}}$, where $A_{\mathscr{R}} \geqq \delta I$ $(\delta > 0)$, then the operator $A_{\mathscr{R}}$ is invertible, and from the relation

$$A_{\mathscr{R}} + iA_{\mathscr{J}} = A_{\mathscr{R}}^{1/2}(I + iA_{\mathscr{R}}^{-1/2} A_{\mathscr{J}} A_{\mathscr{R}}^{-1/2}) A_{\mathscr{R}}^{1/2}$$

it follows that the operator A is likewise invertible.

Let us clarify the route by which this theorem is obtained. Since $\mathscr{Y}^2 = I$, the equality (12.7) can be rewritten as $\mathscr{Y}\mathscr{A} = \mathscr{A}\mathscr{Y}$. This equality shows that the operator $\mathscr{A}$ is selfadjoint with respect to the indefinite scalar product $[x,y] = (\mathscr{Y}x,y)$ $(x,y \in \tilde{\mathfrak{H}})$. This circumstance, together with the complete continuity of the operator $\mathscr{A}_{\mathscr{I}}$, allows us to apply to the operator $\mathscr{A}$ the generalization of a well-known theorem of Pontrjagin, obtained by H. Langer [**3**] (cf. also M. G. Kreĭn [**11**]). From this one deduces the existence of an operator $Z_+ \in \mathfrak{S}_\infty$ such that

$$Z_+^2 + BZ_+ + C = 0, \qquad Z_+^* Z_+ \leqq C, \tag{12.10}$$

and having the following spectral properties: the set of all nonreal characteristic numbers of the operator Z_+ coincides with the set of all characteristic numbers of the bundle L, lying in the upper halfplane, and for every point λ of this set the root lineals $\mathfrak{L}_{1/\lambda}(Z_+)$ and $\mathfrak{L}_\lambda(L)$ coincide.

From the second of the relations (12.10) it follows that

$$s_j(Z_+) \leqq \sqrt{\lambda_j(C)} \qquad (j = 1, 2, \cdots). \tag{12.11}$$

Thus the first assertion of Theorem 12.1 is obtained from the indicated spectral properties of the operator Z_+ and the fact that $Z_+ \in \mathfrak{S}_\infty$.

The second assertion, i.e. the relations (12.9), are obtained by applying Lemma II.3.4 to the operator Z_+, taking the bounds (12.11) into account.

4. If $B \geqq 0$, the operator $\mathscr{A}$ is *R-dissipative*, i.e. $\operatorname{Re}\mathscr{A} \geqq 0$, and consequently the operator $i\mathscr{A}$ is simply dissipative, and Theorems 2.1 and 4.1 are applicable to it.

Noting that

$$\operatorname{sp} B = -\operatorname{sp} \mathscr{A}_{\mathscr{R}},$$

and considering the relations (12.6) and the relation between the operator $\mathscr{A}$ and the bundle $L(\lambda)$, we arrive at the following result.

THEOREM 12.2. *Let* $B \geqq 0$ *and* $\operatorname{sp} B < \infty$. *Then*

$$-\sum \operatorname{Re}(1/\lambda_j) \leqq \operatorname{sp} B, \tag{12.12}$$

where the summation is extended over all characteristic numbers λ_j *of the bundle* L *(taking into account their algebraic multiplicities). The equal sign in* (12.12) *holds if and only if the system of eigenvectors and associated vectors of the bundle* L *is two-fold complete. This two-fold completeness holds whenever* $\underline{\lim}\, n^2 \lambda_n(C) = 0$, *in particular when* $\operatorname{sp}(C^{1/2}) < \infty$.

In connection with the relations (12.6) we remark that Theorem II.6.1, being applicable to the operator $\mathscr{A}$, enables us to make the following general assertion.

If $B = B^* \in \mathfrak{S}_\omega$ then

$$\sum_{j=1}^{n} \left| \operatorname{Re} \frac{1}{\lambda_j'(L)} \right| \leqq \sum_{j=1}^{n} s_j(B) \qquad (n = 1, 2, \cdots, \omega),$$

where $\{\lambda_j'(L)\}_1^\omega$ $(\omega \leqq \infty)$ is the complete sequence of all the characteristic numbers of the bundle L, not lying on the imaginary axis, and ordered according to the absolute value of their real parts: $|\operatorname{Re}\lambda_1'(L)| \leqq |\operatorname{Re}\lambda_2'(L)| \leqq \cdots$.

5. If $B = 0$ the spectrum $\sigma(L)$ lies on the imaginary axis. We shall show that for a "small" (in a specified sense) operator $B = B^*$ the spectrum $\sigma(L)$ will lie in small vertical sectors containing the imaginary axis; here certain additional conditions regarding the operator C will guarantee the two-fold completeness of the system of eigenvectors and associated vectors of the bundle.

THEOREM 12.3. *Let $B = B^*$, and suppose that for some κ $(0 < \kappa < 2)$ one has $B^2 \leqq \kappa^2 C$. Then the spectrum $\sigma(L)$ lies in the vertical sectors*

$$\left| \arg\lambda \pm \frac{\pi}{2} \right| < \theta_1, \tag{12.13}$$

where $\sin\theta_1 = \kappa/2$ $(0 < \theta_1 < \pi/2)$. If $0 < \kappa \leqq 1$ and moreover

$$\lambda_n(C) = o(n^{-2\theta/\pi}), \tag{12.14}$$

where $\sin\theta = \kappa$ $(0 < \theta < \pi/2)$, then the system of eigenvectors and associated vectors of the bundle L is two-fold complete.

PROOF. Suppose that for some complex λ and $\phi \in \mathfrak{H}$ $(|\phi| = 1)$ we have $L(\lambda)\phi = 0$. Then, solving the quadratic equation $(L(\lambda)\phi, \phi) = 0$ with respect to λ, we find that

$$\lambda^{-1} = \tfrac{1}{2}\left(-(B\phi,\phi) \pm i\sqrt{4(C\phi,\phi) - (B\phi,\phi)^2}\right).$$

Since

$$(B\phi,\phi)^2 \leqq |B\phi|^2|\phi|^2 = (B^2\phi,\phi) \leqq \kappa^2(C\phi,\phi),$$

it follows that

$$\operatorname{Im}(1/\lambda) \geqq \tfrac{1}{2}\sqrt{4 - \kappa^2}\,(C\phi,\phi)^{1/2},$$

$$|\operatorname{Re}(1/\lambda)| \leqq \tfrac{1}{2}\,\kappa(C\phi,\phi)^{1/2},$$

from which

$$|\operatorname{Re}\lambda/\operatorname{Im}\lambda| \leqq \kappa/2\sqrt{1-(\kappa/2)^2}.$$

The first assertion of the theorem follows.

Let us consider the operator

$$\mathscr{A} = -\mathscr{A}^2 = \begin{pmatrix} C - B^2 & -BC^{1/2} \\ C^{1/2}B & C \end{pmatrix}. \tag{12.15}$$

For this operator, for any $x = x_1 \oplus x_2 \in \tilde{\mathfrak{H}}$,

$$\begin{aligned} \operatorname{Re}(\mathscr{A}x, x) &= ((C - B^2)\, x_1, x_1) + (Cx_2, x_2) \\ &\geqq (1-\kappa^2)(Cx_1, x_1) + (Cx_2, x_2) \\ &\geqq 2\sqrt{(1-\kappa^2)(Cx_1, x_1)(Cx_2, x_2)} \end{aligned} \tag{12.16}$$

and

$$\begin{aligned} |\operatorname{Im}(\mathscr{A}x, x)| &= 2|\operatorname{Im}(C^{1/2}Bx_1, x_2)| \leqq 2|(Bx_1, C^{1/2}x_2)| \\ &\leqq 2\sqrt{(B^2x_1, x_1)(Cx_2, x_2)} \leqq 2\kappa\sqrt{(Cx_1, x_1)(Cx_2, x_2)}. \end{aligned} \tag{12.17}$$

Thus for all positive $\kappa \leqq 1$ the operator $\mathscr{A}$ is R-dissipative, and for $\kappa < 1$

$$|\operatorname{Im}(\mathscr{A}x, x)|/\operatorname{Re}(\mathscr{A}x, x) \leqq \kappa/\sqrt{1-\kappa^2} \qquad (x_1, x_2 \neq 0).$$

Consequently

$$0 \leqq |\arg(\mathscr{A}x, x)| \leqq \theta. \tag{12.18}$$

On the other hand, it follows from (12.17) that

$$|\operatorname{Im}(\mathscr{A}x, x)| = |(\mathscr{A}_{\mathscr{I}}x, x)| \leqq \kappa((Cx_1, x_1) + (Cx_2, x_2)),$$

i.e.

$$-\mathscr{C} \leqq \mathscr{A}_{\mathscr{I}} \leqq \mathscr{C}, \quad \text{where } \mathscr{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

From (12.15) it follows at once that $O \leqq \mathscr{A}_{\mathscr{R}} \leqq \mathscr{C}$. Therefore if C, and consequently $\mathscr{C}$, satisfies the condition (12.14), then the operators $\sqrt{(\mathscr{A}_{\mathscr{I}})^2}$ and $\mathscr{A}_{\mathscr{R}}$ will satisfy this condition; hence

$$s_n(\mathscr{A}) = o(n^{-2\theta/\pi}) \qquad (n \to \infty).$$

Thus the operator $\mathscr{A}$ satisfies the conditions of Theorem 6.1 regarding the completeness of the system of root vectors. Since $\mathscr{A}^2 = -\mathscr{A}$,

there follows the completeness of the system of root vectors of the operator $\mathscr{A}$, and hence the second assertion of the theorem is also proved.

6. It is interesting to compare the basic (second) assertion of Theorem 12.3 with what is obtained by applying to the bundle $L(\lambda)$ the general results of §9.

According to the theorem of M. V. Keldyš and its generalization, given by Ju. A. Palant, one can assert (regardless of whether or not $B = B^*$) that *the system of eigenvectors and associated vectors of the bundle* $L(\lambda) = I + \lambda B + \lambda^2 C$ *is two-fold complete whenever* $B = TC^{1/2}$ $(T \in \mathfrak{S}_\infty)$ *and at least one of the operators* T, C *has finite order.*

Whenever these conditions are fulfilled the entire spectrum of the bundle L, with the possible exception of a finite number of points, will lie in the vertical sectors

$$|\arg \lambda \pm \pi/2| < \epsilon,$$

for any arbitrarily small ϵ.

On the other hand, the condition $B^2 \leqq \kappa^2 C$ means that $|Bf| \leqq \kappa |C^{1/2}f|$ $(f \in \mathfrak{H})$ which, in turn, is equivalent to $B = TC^{1/2}$, where $T \in \mathfrak{R}$ and $|T| \leqq \kappa$.

Thus if the conditions of Theorem 12.3 are fulfilled then $B = B^* = TC^{1/2}$, where C has finite order ($p(C) < \infty$) and T is a *bounded* operator with a sufficiently small norm (to a degree determined by the quantity $p(C)$).

7. Theorem 11.1 enables us to establish the following asymptotic result for the bundle L being considered.

THEOREM 12.4. *Let* $B = TC^{1/2}$, *where* $T \in \mathfrak{S}_\infty$, *and suppose that there corresponds to the operator* C *a function* $\phi(r)$ $(0 < r < \infty)$ *such that*

$$\frac{\phi(s)}{\phi(r)} \leqq \left(\frac{s}{r}\right)^\gamma \qquad (0 < r < s < \infty, \gamma \text{ a constant})$$

and

$$\lim_{r \to \infty} \frac{n(r; C)}{\phi(r)} = 1.$$

Then for the distribution function $n_+(r; L)$ *of the set of characteristic numbers, lying in the upper halfplane* $\operatorname{Im}\lambda > 0$, *of the bundle* L *we have the following asymptotic relation:*

$$\lim_{r \to \infty} \frac{n_+(\sqrt{r}; L)}{\phi(r)} = 1. \tag{12.19}$$

PROOF. From $B = TC^{1/2}$ it follows that $B = B^* = C^{1/2}T^*$; hence, for the case being considered, the equality (12.15) assumes the form

$$\mathscr{A} = \begin{pmatrix} C - TCT^* & -TC \\ CT & C \end{pmatrix} = (I - \mathscr{S})\mathscr{C}(I + \mathscr{S}^*),$$

where

$$\mathscr{S} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

Obviously $\mathscr{S} \in \mathfrak{S}_\infty$. Since $\mathscr{S}^2 = \mathscr{S}^{*2} = 0$, the operators $I - \mathscr{S}$ and $I + \mathscr{S}^*$ are invertible. Taking into consideration that $n(r; \mathscr{C}) = 2n(r; C)$, we conclude that Theorem 11.1, in the extended formulation indicated in Remark 11.1, is applicable to the operator $\mathscr{A}$. Thus

$$\lim_{r \to \infty} \frac{n(r; \mathscr{A})}{2\phi(r)} = 1, \tag{12.20}$$

and since $n(r; \mathscr{A}) = n(r; \mathscr{A}^2) = n(\sqrt{r}; \mathscr{A}) = n(\sqrt{r}; L) = 2n_+(\sqrt{r}; L)$, (12.19) follows from (12.20). The theorem is proved.

8. The bundle $L(\lambda)$ is said to be *weakly damped* if

$$(Bf, f)^2 < 4(f, f)(Cf, f) \qquad (f \in \mathfrak{H}, f \neq 0). \tag{12.21}$$

When condition (12.21) is fulfilled the selfadjoint operator B belongs to $\mathfrak{S}_\infty$. In fact, let P_n $(n = 1, 2, \cdots)$ be the orthoprojector onto the linear hull of the first n eigenvectors of the operator C, so that

$$|Q_n C Q_n| \leqq \lambda_{n+1}(C), \quad \text{where} \quad Q_n = I - P_n \qquad (n = 1, 2, \cdots).$$

It follows from (12.21) that for any $f \in \mathfrak{H}$ with $|f| = 1$

$$(BQ_n f, Q_n f)^2 \leqq 4(Q_n f, Q_n f)\, \lambda_{n+1}(C) \leqq 4\lambda_{n+1}(C),$$

from which

$$|Q_n B Q_n| = |B - BP_n - P_n B - P_n B P_n| \leqq 4\lambda_{n+1}(C) \to 0 \qquad (n \to \infty)$$

and so $B \in \mathfrak{S}_\infty$.

The condition (12.21) means that for any $f \neq 0$ $(f \in \mathfrak{H})$ the quadratic (in λ) equation $(L(\lambda)f, f) = 0$ does not have real roots, and so $L(\lambda)f \neq 0$ for $f \neq 0$ and $\operatorname{Im} \lambda = 0$.

Thus if the bundle L is weakly damped, then its entire spectrum consists of isolated nonreal characteristic numbers of finite algebraic mul-

tiplicity, with only one possible limit point at infinity.

It can be shown that this result admits a converse.

This fact, in particular, is in agreement with the fact that if $B^2 \leqq \kappa^2 C$ $(0 < \kappa < 2)$ (and consequently the spectrum $\sigma(L)$ lies in the sectors (12.13)), then the condition (12.21) is fulfilled. Indeed, in this case

$$(Bf,f)^2 \leqq (B^2f,f)(f,f) \leqq \kappa^2(Cf,f)(f,f) < 4(Cf,f)(f,f)(|f| \neq 0).$$

We remark further that for commuting B and C the condition (12.21) is equivalent to $B^2 < 4C$, which we have already encountered (cf. §12.3).

9. Let us introduce certain new definitions. Let ϕ_0 be some eigenvector of the bundle: $L(\lambda_0)\phi_0 = 0$. Then three cases are possible: the quantity $|\lambda_0|^2$ can be equal to, greater than or less than the ratio $(\phi_0, \phi_0)/(C\phi_0, \phi_0)$. Corresponding to these cases the eigenvector ϕ_0 will be called *neutral, of the first kind* or *of the second kind.*

Since λ_0 is a root of the quadratic equation

$$((L(\lambda)\,\phi_0, \phi_0) \equiv)\,(C\phi_0, \phi_0)\,\lambda^2 + (B\phi_0, \phi_0)\,\lambda + (\phi_0, \phi_0) = 0,$$

for nonreal λ_0 we always have $|\lambda_0|^2 = (\phi_0, \phi_0)/(C\phi_0, \phi_0)$. Thus the eigenvectors of the bundle L corresponding to nonreal characteristic numbers are always neutral.

If all the eigenvectors of the bundle, corresponding to a given characteristic number λ_0, are of the same kind (first or second), then L has no associated vectors corresponding to the number λ_0. In this case the characteristic number is said to be *definite* and, according to the case involved, a *characteristic number of the first kind* or *second kind.* The bundle L is said to be *strongly damped* if

$$(Bf,f) > 2\sqrt{(f,f)(Cf,f)} \qquad (f \in \mathfrak{H}, f \neq 0). \tag{12.22}$$

This condition indicates that for any $f \neq 0$ the quadratic equation $(L(\lambda)f,f) = 0$ has two distinct negative real roots. It is easy to conclude from this, that:[38)]

1. *Every characteristic number of a strongly damped bundle is negative and definite.*

It is somewhat more difficult to the prove the next assertion:

2. *To any strongly damped bundle L there corresponds a number* $l > 0$

[38)] In the algebraic case ($\mathfrak{H}$ finite-dimensional) results 1 and 2 were established by the American worker in theoretical mechanics R. J. Duffin [1] in his original work, devoted to strongly damped oscillatory systems with a finite number of degrees of freedom.

such that

$$(Bf,f) \geqq (1/l)\,(f,f) \qquad (f \in \mathfrak{H}) \tag{12.23}$$

and every characteristic number of the first kind of the bundle L will be $\leqq -l$, and of the second kind, $\geqq -l$ (and <0).

The relation (12.23) shows that for a strongly damped bundle L the operator B is always *uniformly positive,*[39] and therefore $B \notin \mathfrak{S}_\infty$. This circumstance enables us to make the following addition to result 2.

3. *The spectrum $\sigma(L)$ of a strongly damped bundle consists of characteristic numbers of the first and second kind of finite algebraic multiplicity, and those negative numbers λ $(>-l)$ for which the number $-\lambda^{-1}$ belongs to the condensed spectrum of B.*

To prove the second part of the following result requires rather subtle arguments.

4. *The set of characteristic numbers of the first kind of a strongly damped bundle L can be arranged in a complete nonincreasing sequence $\{\lambda_n^{(1)}(L)\}$ which tends to $-\infty$. One can associate with this sequence a sequence of eigenvectors $\{\phi_n^{(1)}\}$ of the bundle L which forms a Riesz basis*[40] *of the space $\mathfrak{H}$.*

If the condensed spectrum of B consists of a single point β (>0), i.e.,

$$B = \beta I + T \qquad (T \in \mathfrak{S}_\infty), \tag{12.24}$$

result 4 can be supplemented by the following result.

5. *If the condition* (12.24) *is fulfilled, then the spectrum $\sigma(L)$ of the strongly damped bundle L consists of a complete nonincreasing sequence $\{\lambda_n^{(1)}(L)\}$ of characteristic numbers of the first kind, tending to $-\infty$, a complete nonincreasing sequence $\{\lambda_n^{(2)}(L)\}$ of characteristic numbers of the second kind, tending to $-\beta$, and the number $-\beta$ itself. One can associate with the sequences $\{\lambda_n^{(1)}(L)\}$ and $\{\lambda_n^{(2)}(L)\}$ sequences of eigenvectors*

$$\{\phi_n^{(1)}\} \quad \textit{and} \quad \{\phi_n^{(2)}\} \qquad (L(\lambda_n^{(k)})\,\phi_n^{(k)} = 0, \quad n = 1,2,\cdots; k = 1,2),$$

each of which forms a Riesz basis of the space $\mathfrak{H}$. The union of the sequences $\{\phi_n^{(1)}\}$ and $\{\phi_n^{(2)}\}$ yields a two-fold complete system of eigenvectors of the bundle L. For the sequence $\{\lambda_n^{(1)}(L)\}$ we have the asymptotic formula

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[39] An operator B is said to be *uniformly positive* if $B \geqq \delta I$, where $\delta > 0$.

[40] Regarding the notion of a Riesz basis, see Chapter VI §2.

$$\lambda_n^{(1)}(L) = -\frac{\beta}{\lambda_n(C)}(1 + o(1)) \qquad (n \to \infty).$$

It turns out that the second assertion of this result can be strengthened as follows. Suppose that the operators $C^{1/2}$ and T belong to the same s.n. ideal $\mathfrak{S}$. Then there corresponds to the strongly damped bundle L an operator $S \in \mathfrak{S}$ and two orthonormal bases $\{\psi_n^{(1)}\}$ and $\{\psi_n^{(2)}\}$ of the space such that the operator $I + S$ carries them into complete sequences $\{\phi_n^{(1)}\}$ and $\{\phi_n^{(2)}\}$ of eigenvectors of the first and second kind, respectively, of the bundle L. In particular, if $\operatorname{sp} C < \infty$ and $T \in \mathfrak{S}_2$, then the bases $\{\phi_n^{(1)}\}$ and $\{\phi_n^{(2)}\}$ will be Bari bases (bases which are quadratically close to orthonormal bases; cf. Chapter VI, §3).

We remark that result 5 admits certain generalizations to the case in which the operator B has the form (12.24), and the condition (12.22) of strong damping is not fulfilled.

Moreover, all of the results 1—4 admit generalizations to the case in which the operator $B = B^*$ has a strictly positive condensed spectrum and the condition (12.24) is not fulfilled. We note that in this general case the spectrum $\sigma(L)$ contains at most a finite number of nonreal characteristic numbers and, more generally, at most a finite number of characteristic numbers to which correspond neutral eigenvectors, and all of these numbers have finite algebraic multiplicity.

10. Recently S. G. Kreĭn [1] showed that the problem of small oscillations of a viscous fluid, lying in a fixed vessel and having a free surface, leads to the equation

$$y = \mu G y + (1/\mu) H y, \tag{12.25}$$

where $G, H \in \mathfrak{S}_\infty$, $G > 0$, $H \geqq 0$, and μ is a complex parameter.

The substitution $\mu = -\lambda - a$ $(a > 0)$ transforms the equation (12.25) into the following:

$$(a^2 G + aI + H)\, y + \lambda(2aG + I)\, y + \lambda^2 G y = 0.$$

For any operator $H = H^*$ $(\in \mathfrak{R})$ (neither of the conditions $H \in \mathfrak{S}_\infty$, $H \geqq 0$ is necessary) the operator $F = a^2 G + H + aI$ will be uniformly positive for sufficiently large a. Choosing such an a and making the change of variables $x = F^{1/2} y$ in the equation (12.25), we transform it into the equation $L_a(\lambda)\, x = 0$, where

$$L_a(\lambda) = I + \lambda B_a + \lambda^2 C_a, \qquad B_a = F^{-1/2}(2aG + I)\, F^{-1/2}$$

$$C_a = F^{-1/2} G F^{-1/2}.$$

If $H \in \mathfrak{S}_\infty$, then $F - aI \in \mathfrak{S}_\infty$; hence $F^{1/2} - a^{1/2} I \in \mathfrak{S}_\infty$ and so $B_a - a^{-1} I \in \mathfrak{S}_\infty$.

The bundle L_a will be strongly damped if and only if[41]

$$4\,(Gx,x)\,(Hx,x) < (x,x)^2 \qquad (x \in \mathfrak{H},\, x \neq 0). \tag{12.26}$$

In fact, condition (12.26) for the bundle L_a indicates that for any $f \neq 0$ the quadratic equation $(L_a(\lambda)\,f,f) = 0$ has two distinct real roots, and this property is invariant under all the transformations which bring us back to the bundle $H - \lambda I + \lambda^2 G$.

If condition (12.26) is fulfilled, all of the preceding theory is applicable to the equation (12.25). If, moreover, $H \in \mathfrak{S}_\infty$, then the operator B_a satisfies the condition (12.24) with $\beta = 1/a$, and $\lambda_n(C_a)/\lambda_n(G) \to 1/a$ (by virtue of Theorem 11.3).

All this enables us to obtain a number of essential additions to the work of Askerov, Kreĭn and Laptev [1]. For example, in the case where the condition (12.26) and the conditions G, $H \in \mathfrak{S}_\infty$, $G > 0$ are fulfilled, one can assert the existence for equation (12.25) of two Riesz bases consisting respectively of eigenvectors of (12.25) of the first and second kind,[42] and that for the corresponding complete sequence $\mu_1^{(1)} \leqq \mu_2^{(1)} \leqq \cdots$ of characteristic numbers of the first kind of (12.25) one has the asymptotic formula

$$\mu_n^{(1)} = \lambda_n^{-1}(G)\,(1 + o(1)) \qquad (n \to \infty),$$

and if $H > 0$, then also for the corresponding sequence of characteristic numbers of the second kind one has

$$\mu_n^{(2)} = \lambda_n(H)\,(1 + o(1)) \qquad (n \to \infty).$$

Thus the asymptotic behavior of the characteristic numbers of the first kind are determined by the characteristic numbers of the equation $y = \mu G y$, and the asymptotic behavior of the characteristic numbers of the second kind, by the eigenvalues of the equation $Hy = \lambda y$.

Under the asumptions G, $H \in \mathfrak{S}_\infty$, $G > 0$ one can also assert the twofold completeness of all the eigenvectors of the bundle (12.25). However,

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[41] We remark that for $H \leqq 0$ inequality (12.26) is automatically fulfilled. If $H = H_+ - H_-$ $(H_\pm \geqq 0)$, then the inequality (12.26) will be fulfilled, for example, if $4\,|G|\,|H_+| < 1$.

[42] It is not hard to see how the concept of an eigenvector (characteristic number) of the first or second kind is carried over to the equation (12.25).

G. I. Laptev has indicated an ingenious method which makes it possible, under very general conditions, to transform the equation (12.25) into a form to which Keldyš' criterion for two-fold completeness is applicable.

The equation (12.25) can be rewritten in each of the following two forms:

$$y + \frac{1}{\mu}(G - H)y = \left(\mu + \frac{1}{\mu}\right) Gy,$$

$$(G - H)y + \frac{1}{\mu} y = \left(\mu + \frac{1}{\mu}\right) \frac{1}{\mu} Hy.$$

The substitutions $z = y/\mu$ and $\lambda = \mu + 1/\mu$ reduce these equations to the form

$$y + (G - H)z = \lambda Gy, \qquad (G - H)y + z = \lambda Hz.$$

In operator form, this system can be written in the form of a linear bundle

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -G + H \\ -H + G & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \lambda \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

to which Keldyš' theorem on the two-fold completeness of the eigenvectors and associated vectors is applicable, in the case where the operators G and H are complete and have finite orders. We remark that G. I. Laptev's transformation is also applicable to equations of a more general type:

$$y = Ty + \mu Gy + (1/\mu) Hy, \tag{12.27}$$

where $T \in \mathfrak{S}_\infty$. In this case the transformed equation will have the form

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} T & H - G \\ G - H & T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \lambda \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

CHAPTER VI

BASES. TESTS FOR THE EXISTENCE OF BASES, CONSISTING OF ROOT VECTORS OF A DISSIPATIVE OPERATOR

As soon as one has established the completeness of the root vectors of some nonselfadjoint operator, there immediately arises the question of whether one can form a basis from the root vectors of this operator.

There are only a few investigations in this direction. The most simply formulated results concern dissipative operators with completely continuous imaginary component. This chapter is devoted to them.

It turns out that if the spectrum of a dissipative operator of the type indicated is sufficiently "compressed" towards the real axis, then one can form, from the eigenvectors of this operator, a basis for their closed linear hull, and hence, under specified conditions, a basis for the entire space.

In accordance with the degree to which this spectrum is "compressed" towards the real axis, the indicated basis can turn out to be a basis of one kind or another in the sense of its closeness to an orthonormal basis.

In view of the absence of textbooks in which one might find a discussion of the basic theory of bases in a Hilbert space, we have considered it advisable to devote several sections to the discussion of bases.

In putting together this chapter we have been aided substantially by A. S. Markus, particularly in those parts of §§5 and 6 where results of his are discussed.

§1. Bases of a Hilbert space

1. A sequence $\{\phi_j\}_1^\infty$ of vectors of a Banach space $\mathfrak{B}$ is called a *basis* of this space if every vector $x \in \mathfrak{B}$ can be expanded in a unique way in a series

$$x = \sum_{j=1}^{\infty} c_j \phi_j, \tag{1.1}$$

which converges in the norm of the space $\mathfrak{B}$. In this expansion the coefficients c_j are obviously linear functionals of the elcment $x \in \mathfrak{B}$:

$$c_j = \Psi_j(x) \qquad (j = 1, 2, \cdots). \tag{1.2}$$

Moreover, by a well-known theorem of Banach (see Banach [1] or

Ljusternik and Sobolev [1]), these linear functionals are continuous ($\Psi_j \in \mathfrak{B}^*$; $j = 1, 2, \cdots$) and there exists a constant C_ϕ associated with them such that

$$(1.3) \qquad |\phi_j|^{-1} \leqq |\Psi_j| \leqq C_\phi |\phi_j|^{-1}.$$

We shall apply these general results to a basis $\{\phi_j\}$ of a Hilbert space $\mathfrak{B} = \mathfrak{H}$. In this case the relations (1.2) can be written in the form

$$(1.4) \qquad c_j = (x, \psi_j) \qquad (\psi_j \in \mathfrak{H}; j = 1, 2, \cdots).$$

Setting $x = \phi_k$ ($k = 1, 2, \cdots$), we obtain

$$(\phi_k, \psi_j) = \delta_{jk} \qquad (j, k = 1, 2, \cdots).$$

Let us recall that two sequences $\{\chi_j\}$ and $\{\omega_j\}$ with elements from $\mathfrak{H}$ are said to be *biorthogonal*, if

$$(\chi_j, \omega_k) = \delta_{jk} \qquad (j, k = 1, 2. \cdots).$$

For a given sequence $\{\chi_j\}_1^\infty \in \mathfrak{H}$ a biorthogonal sequence $\{\omega_j\}_1^\infty \in \mathfrak{H}$ exists if and only if each element χ_j ($j = 1, 2, \cdots$) lies outside the closed linear hull $\mathfrak{L}_j$ of all the other elements χ_k ($k \neq j$). If this condition is fulfilled then the biorthogonal sequence $\{\omega_j\}_1^\infty$ will be uniquely determined if and only if the system $\{\chi_j\}_1^\infty$ is complete in $\mathfrak{H}$. In this case the orthogonal complement $\mathfrak{L}_j^\perp = \mathfrak{H} \ominus \mathfrak{L}_j$ ($j = 1, 2, \cdots$) is one-dimensional, and the element ω_j is determined by the conditions $\omega_j \in \mathfrak{L}_j^\perp$, $(\omega_j, \chi_j) = 1$ ($j = 1, 2, \cdots$).

Thus for every basis $\{\phi_j\}_1^\infty$ the biorthogonal sequence $\{\psi_j\}_1^\infty$ is defined uniquely.

From the equalities (1.1) and (1.4) it follows that any vector f which is orthogonal to all the vectors ψ_j ($j = 1, 2, \cdots$) equals zero. Consequently the sequence biorthogonal to a basis is always complete in $\mathfrak{H}$.

THEOREM 1.1 (S. BANACH [1]). *The sequence $\{\psi_j\}_1^\infty$ biorthogonal to a basis $\{\phi_j\}_1^\infty$ of a Hilbert space $\mathfrak{H}$ is also a basis of $\mathfrak{H}$.*

PROOF. Any vector $f \in \mathfrak{H}$ can be expanded in a series

$$f = \sum_{j=1}^{\infty} (f, \psi_j) \phi_j$$

which is convergent in the norm, and so for any $\chi \in \mathfrak{H}$ the numerical series

$$(f, \chi) = \sum_{j=1}^{\infty} (f, \psi_j)(\phi_j, \chi)$$

converges. Thus

$$(f, \chi) = \lim_{n \to \infty} \left(f, \sum_{j=1}^{n} (\chi, \phi_j) \psi_j \right) \qquad (f \in \mathfrak{H}).$$

This means that the sequence of projectors $\{Q_n\}_1^\infty$ defined by

(1.5) $$Q_n \chi = \sum_{j=1}^{n} (\chi, \phi_j) \psi_j \qquad (n = 1, 2, \cdots; \chi \in \mathfrak{H}),$$

converges weakly to the unit operator. From this it follows (see Ahiezer and Glazman [1]) that the sequence of projectors $\{Q_n\}$ is bounded:

$$\sup_n |Q_n| = C < \infty.$$

By virtue of the completeness of the sequence $\{\psi_j\}_1^\infty$ in $\mathfrak{H}$, for any $\epsilon > 0$ and vector $\chi \in \mathfrak{H}$ one can find numbers $c_j^{(\epsilon)}$ $(j = 1, 2, \cdots, N_\epsilon)$ such that

(1.6) $$|\chi - \sum_{j=1}^{N_\epsilon} c_j^{(\epsilon)} \psi_j| < \epsilon,$$

and so, for the vector

$$Q_n \left(\chi - \sum_{j=1}^{N_\epsilon} c_j^{(\epsilon)} \psi_j \right) = Q_n \chi - \sum_{j=1}^{N_\epsilon} c_j^{(\epsilon)} \psi_j \qquad (n > N_\epsilon)$$

we have

(1.7) $$|Q_n \chi - \sum_{j=1}^{N_\epsilon} c_j^{(\epsilon)} \psi_j| < C\epsilon \qquad (n > N_\epsilon).$$

It follows from (1.6) and (1.7) that for $n > N_\epsilon$

$$|Q_n \chi - \chi| < (1 + C)\epsilon.$$

Thus any vector $\chi \in \mathfrak{H}$ can be expanded in a series

$$\chi = \sum_{j=1}^{\infty} c_j \psi_j$$

which converges in the norm, and the coefficients c_j $(j = 1, 2, \cdots)$ are uniquely defined by

$$c_j = (\chi, \phi_j) \qquad (j = 1, 2, \cdots).$$

The theorem is proved.

2. We shall say that a sequence $\{\phi_j\}$ of vectors from $\mathfrak{H}$ is *almost normalized* if

$$\inf_n |\phi_n| > 0 \quad \text{and} \quad \sup_n |\phi_n| < \infty.$$

1. *If the basis* $\{\phi_j\}_1^\infty$ *of the space* $\mathfrak{H}$ *is almost normalized, then the biorthogonal basis* $\{\psi_j\}_1^\infty$ *is almost normalized.*

In fact, according to (1.3)

$$\inf_n |\psi_n| \geqq \frac{1}{\sup_n |\phi_n|} > 0$$

and

$$\sup_n |\psi_n| \leqq C_\phi \frac{1}{\inf_n |\phi_n|} < \infty.$$

§2. Bases equivalent to orthonormal bases (Riesz bases)

1. Let $\{\phi_j\}$ be an arbitrary orthonormal basis of the space $\mathfrak{H}$, and A some bounded linear invertible operator. Then for any vector $f \in \mathfrak{H}$ one has

$$A^{-1}f = \sum_{j=1}^\infty (A^{-1}f, \phi_j)\phi_j = \sum_{j=1}^\infty (f, A^{*-1}\phi_j)\,\phi_j,$$

and consequently

$$f = \sum_{j=1}^\infty (f, \chi_j)\psi_j,$$

where

$$\psi_j = A\phi_j, \quad \chi_j = A^{*-1}\phi_j \qquad (j = 1, 2, \cdots). \tag{2.1}$$

Obviously

$$(\psi_j, \chi_k) = \delta_{jk} \qquad (j, k = 1, 2, \cdots).$$

Therefore if

$$f = \sum_{j=1}^\infty c_j\psi_j, \tag{2.2}$$

then

$$c_j = (f, \chi_j) \qquad (j = 1, 2, \cdots),$$

i.e. the expansion (2.2) is unique.

Thus every bounded invertible operator transforms any orthonormal basis into some other basis of the space $\mathfrak{H}$. A basis $\{\psi_j\}$ of the space $\mathfrak{H}$ which is obtained from an orthonormal basis by means of such a transformation is called a basis *equivalent to an orthonormal basis* (in the terminology of N. K. Bari [1], a *Riesz basis*).

Suppose that the bounded invertible operator A transforms the ortho-

normal basis $\{\phi_j\}$ into the basis $\{\psi_j\}$. Then, according to (2.1), the operator A^{*-1} transforms the basis $\{\phi_j\}$ into the basis $\{\chi_j\}$ which is biorthogonal to $\{\psi_j\}_1^\infty$. *Consequently a basis which is biorthogonal to a basis which is equivalent to an orthonormal basis is itself equivalent to an orthonormal basis.*

Since

$$\sup_n |\psi_n| \leqq |A| \quad \text{and} \quad \inf_n |\psi_n| \geqq \frac{1}{|A^{-1}|}, \tag{2.3}$$

every basis which is equivalent to an orthonormal basis is almost normalized.

From this it is easy to deduce that *if the basis $\{\psi_j\}_1^\infty$ is equivalent to an orthonormal basis, then the sequence of unit vectors*

$$\{\hat{\psi}_j\}_1^\infty \quad (\hat{\psi}_j = \psi_j / |\psi_j|;\ j = 1, 2, \cdots)$$

also forms a basis equivalent to an orthonormal basis.

In fact, the relations

$$B\phi_j = \phi_j / |\psi_j| \qquad (j = 1, 2, \cdots)$$

obviously define a bounded linear invertible operator. Consequently, the operator AB is invertible and

$$AB\phi_j = \psi_j / |\psi_j| = \hat{\psi}_j \qquad (j = 1, 2, \cdots).$$

2. We shall formulate a number of characteristic properties of bases equivalent to orthonormal bases.

THEOREM 2.1 (N. K. BARI [2]). *The following assertions are equivalent.*

1) *The sequence $\{\psi_j\}_1^\infty$ forms a basis of the space $\mathfrak{H}$, equivalent to an orthonormal basis.*

2) *The sequence $\{\psi_j\}_1^\infty$ becomes an orthonormal basis of the space $\mathfrak{H}$ following the appropriate replacement of the scalar product (f,g) by some new one $(f,g)_1$, topologically equivalent*[1] *to the original one.*

3) *The sequence $\{\psi_j\}_1^\infty$ is complete in $\mathfrak{H}$, and there exist positive constants a_1, a_2 such that for any positive integer n and any complex numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$ one has*

$$a_2 \sum_{j=1}^n |\gamma_j|^2 \leqq \left| \sum_{j=1}^n \gamma_j \psi_j \right|^2 \leqq a_1 \sum_{j=1}^n |\gamma_j|^2. \tag{2.4}$$

---

[1] The scalar products (f,g) and $(f,g)_1$ are said to be *topologically equivalent* if they generate topologically equivalent norms, i.e. if there exist positive constants c_1, c_2 such that

$$c_1(f,f) \leqq (f,f)_1 \leqq c_2(f,f) \qquad (f \in \mathfrak{H}).$$

4) *The sequence* $\{\psi_j\}_1^\infty$ *is complete in* $\mathfrak{H}$, *and its Gram matrix*

$$\| (\psi_j, \psi_k) \|_1^\infty \tag{2.5}$$

generates a bounded invertible operator in the space l_2.[2)]

5) *The sequence* $\{\psi_j\}_1^\infty$ *is complete in* $\mathfrak{H}$, *there corresponds to it a complete biorthogonal sequence* $\{\chi_j\}_1^\infty$, *and for any* $f \in \mathfrak{H}$ *one has*

$$\sum_{j=1}^{\infty} |(f, \psi_j)|^2 < \infty, \quad \sum_{j=1}^{\infty} |(f, \chi_j)|^2 < \infty.$$

PROOF. Assertion 1) implies 2). In fact, let A be a bounded linear invertible operator which carries the basis $\{\psi_j\}_1^\infty$ into some orthonormal basis $\{\phi_j\}_1^\infty$. Then it is easily seen that the scalar product

$$(f,g)_1 = (Af, Ag)$$

is topologically equivalent to the original one, and

$$(\psi_j, \psi_k)_1 = (A\psi_j, A\psi_k) = (\phi_j, \phi_k) = \delta_{jk} \qquad (j, k = 1, 2, \cdots).$$

Suppose that the sequence $\{\psi_j\}_1^\infty$ becomes an orthonormal basis of the space $\mathfrak{H}$ if the original scalar product in $\mathfrak{H}$ is replaced by some topologically equivalent scalar product $(f,g)_1$. Then from the relations

$$c_1(f,f) \leqq (f,f)_1 \leqq c_2(f,f) \qquad (f \in \mathfrak{H}),$$

where c_1, c_2 are positive constants not depending upon f, it follows that for any complex numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$ $(n = 1, 2, \cdots)$ we have

$$c_2^{-1} \sum_{j=1}^{n} |\gamma_j|^2 \leqq \left| \sum_{j=1}^{n} \gamma_j \psi_j \right|^2 \leqq c_1^{-1} \sum_{j=1}^{n} |\gamma_j|^2.$$

Moreover, the sequence $\{\psi_j\}_1^\infty$ is complete in $\mathfrak{H}$, and consequently assertion 2) implies assertion 3). We shall next show that 3) implies 1).

Let $\{\phi_j\}_1^\infty$ be an arbitrary orthonormal basis of $\mathfrak{H}$. We define operators A and A_1 on the linear hulls of the sequences $\{\phi_j\}_1^\infty$ and $\{\psi_j\}_1^\infty$ respectively, putting

$$A\left(\sum_j \gamma_j \phi_j\right) = \sum_j \gamma_j \psi_j \quad \text{and} \quad A_1\left(\sum_j \gamma_j \psi_j\right) = \sum_j \gamma_j \phi_j.$$

According to (2.4),

$$\left| A\left(\sum_j \gamma_j \phi_j\right) \right| \leqq a_1^{1/2} \left| \sum_j \gamma_j \phi_j \right|$$

and

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2) The equivalence of assertions 1) and 4) was proved independently by R. P. Boas [1].

$$\left| A_1 \left(\sum_j \gamma_j \psi_j \right) \right| \leqq a_2^{\ 1/2} \left| \sum_j \gamma_j \psi_j \right|.$$

Since both sequences $\{\phi_j\}_1^\infty$ and $\{\psi_j\}_1^\infty$ are complete in $\mathfrak{H}$, each of the operators A and A_1 can be extended by continuity to a bounded linear operator defined on the entire space $\mathfrak{H}$. It is easily seen that $AA_1 = A_1A = I$, i.e. the operator A is invertible and $A^{-1} = A_1$. Consequently $\{\psi_j\}$ is a basis equivalent to an orthonormal basis.

Thus the equivalence of the first three assertions is proved.

Let A be a bounded linear invertible operator which carries some orthonormal basis $\{\phi_j\}_1^\infty$ into the basis $\{\psi_j\}_1^\infty$. Since

$$(A^*A\phi_j, \phi_k) = (A\phi_j, A\phi_k) = (\psi_j, \psi_k) \qquad (j, k = 1, 2, \cdots),$$

it follows that to the operator A^*A there corresponds, in the basis $\{\phi_j\}_1^\infty$, the matrix (2.5). Thus 4) follows from 1). We shall show that 4) implies 3).

Suppose that the matrix (2.5) generates a bounded invertible operator in l_2, and let $\{\phi_j\}_1^\infty$ be an arbitrary orthonormal basis of the space $\mathfrak{H}$. Then the operator H defined in $\mathfrak{H}$ by

$$H\left(\sum_j \alpha_j \phi_j\right) = \sum_j \phi_j \sum_{k=1}^\infty (\psi_k, \psi_j)\alpha_k \qquad \left(\sum_j |\alpha_j|^2 < \infty\right)$$

is obviously a bounded linear positive invertible operator.

One can verify without difficulty that for any complex numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$

$$\left|\sum_j \gamma_j \psi_j\right|^2 = \left(H\left(\sum_j \gamma_j \phi_j\right), \sum_j \gamma_j \phi_j\right)$$

or

$$\left|\sum_j \gamma_j \psi_j\right|^2 = \left|H^{1/2}\left(\sum_j \gamma_j \phi_j\right)\right|^2.$$

It follows at once that

$$|H^{-1}|^{-1} \sum_j |\gamma_j|^2 \leqq \left|\sum_j \gamma_j \psi_j\right|^2 \leqq |H| \sum_j |\gamma_j|^2.$$

Consequently 3) follows from 4).

Making use of the auxiliary results established at the beginning of this section, and of the equivalence of the assertions 1) and 3), it is easily deduced that assertion 5) follows from assertion 1). We shall prove, finally, that assertion 5) implies 1). Let us consider the convex functional

$$p(f) = \left(\sum_{j=1}^\infty |(f, \psi_j)|^2\right)^{1/2} \qquad (f \in \mathfrak{H}).$$

Since this functional is the supremum of the continuous convex functionals

$$p_n(f) = \left(\sum_{j=1}^{n} |(f,\psi_j)|^2\right)^{1/2} \qquad (f \in \mathfrak{H}),$$

by a lemma of I. M. Gel′fand (see Ahiezer and Glazman [1]) the functional $p(f)$ is continuous, and consequently there exists a constant $c_1 > 0$ such that

$$p(f) \leqq c_1 |f|$$

or

$$\sum_{j=1}^{\infty} |(f,\psi_j)|^2 \leqq c_1^2 |f|^2. \tag{2.6}$$

One can similarly prove that there exists a constant $c_2 > 0$ for which

$$\sum_{j=1}^{\infty} |(f,\chi_j)|^2 \leqq c_2^2 |f|^2. \tag{2.7}$$

Let $\{\phi_j\}_1^\infty$ be an arbitrary orthonormal basis. We define operators A_2 and A_3 on the linear hulls of the sequences $\{\psi_j\}_1^\infty$ and $\{\chi_j\}_1^\infty$ respectively, setting

$$A_2\left(\sum_j \gamma_j \psi_j\right) = \sum_j \gamma_j \phi_j \quad \text{and} \quad A_3\left(\sum_j \gamma_j \chi_j\right) = \sum_j \gamma_j \phi_j.$$

According to (2.6) and (2.7)

$$\left|A_2\left(\sum_j \gamma_j \psi_j\right)\right| \leqq c_2 \left|\sum_j \gamma_j \psi_j\right|$$

and

$$\left|A_3\left(\sum_j \gamma_j \chi_j\right)\right| \leqq c_1 \left|\sum_j \gamma_j \chi_j\right|.$$

Since both sequences $\{\psi_j\}_1^\infty$ and $\{\chi_j\}_1^\infty$ are complete in $\mathfrak{H}$, each of the operators A_2 and A_3 can be extended by continuity to a bounded linear operator defined on the entire space $\mathfrak{H}$. A direct verification shows that

$$\left(A_2\left(\sum_j \gamma_j' \psi_j\right),\ A_3\left(\sum_k \gamma_k'' \chi_k\right)\right) = \left(\sum_j \gamma_j' \psi_j, \sum_k \gamma_k'' \chi_k\right),$$

and so

$$(A_2 f, A_3 g) = (f,g) \qquad (f,g \in \mathfrak{H})$$

and

$$(A_3^* A_2 f, g) = (f, g) \qquad (f, g \in \mathfrak{H}).$$

It follows that $A_3^* A_2 = I$.

Moreover, the range of the operator A_2 is obviously dense in $\mathfrak{H}$; consequently A_2 is invertible.

The theorem is proved.

3. A basis of the space $\mathfrak{H}$ is said to be a *permutable* (*unconditional*) basis, if for any permutation of its terms it remains a basis of $\mathfrak{H}$.

If the basis $\{\psi_j\}_1^\infty$ is permutable, then obviously the basis $\{\chi_j\}_1^\infty$ biorthogonal to it is likewise permutable.

Every orthonormal basis is permutable. Moreover, it is easily seen that any basis which is equivalent to an orthonormal basis is permutable. This property is characteristic for bases which are equivalent to orthonormal bases. For the proof of this result we need:

LEMMA 2.1 *Let $\{x_n\}_1^\infty$ be a sequence of vectors of a Banach space $\mathfrak{B}$. If the partial sums of each series $\sum x_{n'}$, obtained from the series $\sum_{n=1}^\infty x_n$ by means of a permutation of its terms, form a bounded set, then*

$$\sup_{n, |\epsilon_j| \leq 1} \left| \sum_{j=1}^n \epsilon_j x_j \right| < \infty. \tag{2.8}$$

PROOF. Let f be an arbitrary functional from $\mathfrak{B}^*$. It follows from the hypothesis of the lemma that the partial sums of the series

$$\sum_{j=1}^\infty f(x_j) \tag{2.9}$$

are bounded for any given permutation of its terms. Consequently the series (2.9) is absolutely convergent.

The convex functional

$$p(f) = \sum_{j=1}^\infty |f(x_j)| \qquad (f \in \mathfrak{B}^*)$$

is the supremum of the sequence of continuous convex functionals

$$p_n(f) = \sum_{j=1}^n |f(x_j)|,$$

and consequently, by a lemma of I. M. Gel'fand (see Ahiezer and Glazman [1]) the functional $p(f)$ is continuous, i.e.

$$p(f) \geqq c|f| \qquad (f \in \mathfrak{B}^*),$$

where $c > 0$ is a constant independent of f.

Thus for any positive integer n and arbitrary numbers ϵ_j ($|\epsilon_j| \leqq 1$; $j = 1, 2, \cdots, n$) we have

$$\left| \sum_{j=1}^{n} \epsilon_j f(x_j) \right| \leqq c|f|,$$

and consequently

$$\left| \sum_{j=1}^{n} \epsilon_j x_j \right| \leqq c.$$

The lemma is proved.

LEMMA 2.2. *If the sequence $\{g_j\}$ of vectors from $\mathfrak{H}$ satisfies the hypothesis of Lemma* 2.1, *then*

(2.10) $$\sum_{j=1}^{\infty} |g_j|^2 < \infty.$$

PROOF. For any two vectors $f, h \in \mathfrak{H}$ we can always choose a number ϵ of modulus unity such that

$$|f|^2 + |h|^2 \leqq |f + \epsilon h|^2.$$

In fact, if we put

$$\epsilon = \exp(i \arg(f, h)),$$

then

$$|f + \epsilon h|^2 = |f|^2 + |h|^2 + 2|(f, h)| \geqq |f|^2 + |h|^2.$$

We easily conclude that there exists a sequence of numbers ϵ_j ($|\epsilon_j| = 1$; $j = 1, 2, \cdots$) for which

$$\sum_{j=1}^{n} |g_j|^2 \leqq \left| \sum_{j=1}^{n} \epsilon_j g_j \right|^2 \qquad (n = 1, 2, \cdots).$$

Recalling the preceding lemma, we obtain (2.10). The lemma is proved.

Lemmas 2.1 and 2.2, among other results, were proved by W. Orlicz [1].

THEOREM 2.2 (E. R. LORCH [1]). *In order that a basis $\{\psi_j\}_1^\infty$ of the space $\mathfrak{H}$ be equivalent to an orthonormal basis, it is necessary and sufficient that it be permutable and almost normalized.*

PROOF.[3] The necessity of the hypotheses of the theorem has already been noted. We shall prove their sufficiency.

Let f be an arbitrary vector from $\mathfrak{H}$. Then

$$f = \sum_{j=1}^{\infty} (f, \chi_j)\psi_j, \tag{2.11}$$

where $\{\chi_j\}_1^\infty$ is the basis biorthogonal to $\{\psi_j\}_1^\infty$, and the series (2.11) converges for any permutation of its terms. Hence, according to Lemma 2.2,

$$\sum_{j=1}^{\infty} |(f, \chi_j)|^2 |\psi_j|^2 < \infty.$$

Taking into account that the basis $\{\psi_j\}_1^\infty$ is almost normalized, we obtain

$$\sum_{j=1}^{\infty} |(f, \chi_j)|^2 < \infty.$$

Since the basis $\{\chi_j\}$ biorthogonal to the almost normalized permutable basis $\{\psi_j\}$ also has these properties, we also have

$$\sum_{j=1}^{\infty} |(f, \psi_j)|^2 < \infty.$$

By Theorem 2.1 (see assertion 5)) it follows that the basis $\{\psi_j\}$ is equivalent to an orthonormal basis. The theorem is proved.

4. We formulate two definitions.

Two sequences of vectors $\{g_j\}$ and $\{f_j\}$ are said to be *quadratically close* if

$$\sum_{j=1}^{\infty} |g_j - f_j|^2 < \infty.$$

A sequence of vectors $\{g_j\}$ is said to be *ω-linearly independent*[4] if the equality

$$\sum_{j=1}^{\infty} c_j g_j = 0$$

[3] This theorem, together with some generalizations of it, was independently proved in an ingenious way by I. M. Gel′fand [1]. The possibility of an elementary derivation of Theorem 2.2 by means of Lemma 2.2 was brought to our attention by V. Ja. Kozlov in 1956 (cf. also B. E. Veĭc [1]).

[4] All further assertions in which the concept of ω-linear independence enters remain valid if in the definition of this concept the condition (2.12) is replaced by the condition that not all of the c_j equal zero. Replacing one definition by the other will in certain cases strengthen the assertions, and in other cases weaken them.

is not possible for

$$0 < \sum_{j=1}^{\infty} |c_j|^2 |g_j|^2 < \infty. \tag{2.12}$$

If the sequence $\{g_j\}$ is almost normalized, then condition (2.12) is equivalent to

$$0 < \sum_{j=1}^{\infty} |c_j|^2 < \infty.$$

THEOREM 2.3 (N. K. BARI [2]). *Any ω-linearly independent sequence $\{g_j\}$ which is quadratically close to some basis $\{\psi_j\}_1^\infty$ which is equivalent to an orthonormal basis, is itself a basis equivalent to an orthonormal basis.*

PROOF. Let A be a bounded linear invertible operator which carries some orthonormal basis $\{\phi_j\}_1^\infty$ of the space $\mathfrak{H}$ into the basis $\{\psi_j\}$:

$$A\phi_j = \psi_j \qquad (j = 1, 2, \cdots).$$

We define an operator T, putting

$$T\left(\sum_{j=1}^{\infty} c_j\phi_j\right) = \sum_{j=1}^{\infty} c_j(\psi_j - g_j) \quad \left(\sum_j |c_j|^2 < \infty\right).$$

Obviously T is a bounded linear operator, and

$$|T|^2 \leqq \sum_{j=1}^{\infty} |\psi_j - g_j|^2.$$

Moreover, it follows from

$$\sum_{j=1}^{\infty} |T\phi_j|^2 = \sum_{j=1}^{\infty} |\psi_j - g_j|^2 \qquad (< \infty)$$

that $T \in \mathfrak{S}_2$.

The equation $(A - T)\phi = 0$ has a unique solution, zero. In fact, if $A\phi = T\phi$, then from

$$(A - T)\phi = \sum_j (\phi, \phi_j)\psi_j - \sum_j (\phi, \phi_j)(\psi_j - g_j) = \sum_j (\phi, \phi_j) g_j$$

it follows that

$$\sum_j (\phi, \phi_j) g_j = 0,$$

and so $\phi = 0$ by the ω-linear independence of the sequence $\{g_j\}_1^\infty$.

The operator A is invertible, and the operator $T \in \mathfrak{S}_2$ is completely continuous; since the operator $A - T$ annihilates only the zero vector,

it also is invertible. Bearing in mind the obvious equalities

$$(A - T)\,\phi_j = g_j \qquad (j = 1, 2, \cdots),$$

we conclude that the sequence $\{g_j\}_1^\infty$ is a basis equivalent to an orthonormal one. The theorem is proved.

REMARK 2.1. If we discard the condition of the ω-linear independence of the sequence $\{g_j\}$, then for a sequence $\{g_j\}$ which is quadratically close to a basis equivalent to an orthonormal basis, the following assertions hold.

For each relation

$$\sum_{j=1}^{\infty} c_j g_j = 0 \tag{2.13}$$

always

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty.$$

Among the relations of the form (2.13) one can find a finite number p of linearly independent ones

$$\sum_{j=1}^{\infty} c_{kj} g_j = 0 \qquad (k = 1, 2, \cdots, p), \tag{2.14}$$

in terms of which every other relation (2.13) will be linearly expressible. The number p coincides with the dimension of the orthogonal complement of the closed linear hull $\mathfrak{N}_g$ of the sequence $\{g_j\}$. If the vectors g_{j_k} $(k = 1, 2, \cdots, p)$ are, by virtue of the relations (2.14), linearly expressible in terms of the remaining g_j $(j \neq j_k, k = 1, 2, \cdots, p)$, then the sequence $\{g_j\}$ can be made into a basis equivalent to an orthonormal basis by replacing the vectors g_{j_k} $(k = 1, 2, \cdots, p)$ by any other vectors g'_{j_k} $(k = 1, 2, \cdots, p)$ which form a basis in a subspace which is complementary to $\mathfrak{N}_g$ with respect to $\mathfrak{H}$.

REMARK 2.2 (GOHBERG AND MARKUS [1]). Theorem 2.3 and Remark 2.1 remain valid if in their formulation the condition of the quadratic closeness of the sequences $\{g_j\}$ and $\{\psi_j\}$ is replaced by the following condition: the Gram matrix

$$\| (g_j - \psi_j, g_k - \psi_k) \|_1^\infty$$

generates a completely continuous operator in the space l_2. An assertion is also valid which in a certain sense is the converse of this one: if for any orthonormal basis $\{\phi_j\}$ of the space $\mathfrak{H}$ the sequence $\{a\phi_j + h_j\}_1^\infty$ $(a > 0)$

is either a basis equivalent to an orthonormal basis, or else can be made into such by replacing a finite number of its terms by the same number of other vectors, then the Gram matrix

$$\| (h_j, h_k) \|_1^\infty$$

of the sequence $\{h_j\}_1^\infty$ generates a completely continuous operator in the space l_2.

REMARK 2.3. The class of bases which are equivalent to orthonormal bases is very large, and the problem of constructing at least one normalized basis of the space $\mathfrak{H}$ which is not equivalent to an orthonormal basis proved to be not easy. It was solved by K. I. Babenko [1], who showed that the sequence

$$\{ (\alpha + \tfrac{1}{2}) |x|^\alpha e^{\pi i n x} \}_{n=-\infty}^{\infty} \qquad (-\tfrac{1}{2} < \alpha < \tfrac{1}{2};\ \alpha \neq 0)$$

is such a basis in the space $L_2(-1,1)$. Recently this result was generalized by V. F. Gapoškin [1].

§3. Bases quadratically close to orthonormal bases (Bari bases)[5]

1. From Theorem 2.3 of N. K. Bari it follows in particular that each ω-linearly independent sequence of vectors which is quadratically close to some orthonormal basis of the space $\mathfrak{H}$ is a basis of $\mathfrak{H}$. In the terminology of M. G. Kreĭn [7], such a basis is called a *Bari basis*.

According to the theorem of Bari, every basis which is quadratically close to an orthonormal basis is equivalent to an orthonormal basis. It is obvious that for any permutation of the elements of a basis which is quadratically close to an orthonormal basis, one obtains a basis of the same type.

1. *If $\{\psi_j\}_1^\infty$ is a basis quadratically close to an orthonormal basis, then the sequence $\{\hat\psi_j\}_1^\infty$ ($\hat\psi_j = \psi_j / |\psi_j|$; $j = 1, 2, \cdots$) is also a basis quadratically close to an orthonormal one.*

In fact, if $\{\phi_j\}_1^\infty$ is an orthonormal basis quadratically close to $\{\psi_j\}_1^\infty$, then putting $\epsilon_j = \exp i\theta_j$ ($j = 1, 2, \cdots$), where $\theta_j = \arg(\hat\psi_j, \phi_j)$ ($j = 1, 2, \cdots$), we obtain

$$|\hat\psi_j - \epsilon_j \phi_j|^2 = 2(1 - |(\hat\psi_j, \phi_j)|) \leqq 2(1 - |(\hat\psi_j, \phi_j)|^2) \qquad (j = 1, 2, \cdots).$$

Thus

$$|\hat\psi_j - \epsilon_j \phi_j|^2 \leqq 2(1 - |(\hat\psi_j, \phi_j)|^2) = 2 \min_t |\phi_j - t\psi_j|^2 \leqq 2|\phi_j - \psi_j|^2,$$

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[5] In this section we discuss, in a somewhat modified form, the content of a paper by M. G. Kreĭn [7].

and so the sequence $\{\tilde{\psi}_j\}_1^\infty$ is quadratically close to the orthonormal basis $\{\epsilon_j\phi_j\}_1^\infty$.

It is easily seen that among the bases which are equivalent to orthonormal bases, the Bari bases are distinguished by the following test.

2. *In order that a sequence $\{\psi_j\}_1^\infty$ be a basis of the space $\mathfrak{H}$ quadratically close to an orthonormal basis, it is necessary and sufficient that there exist an orthonormal basis $\{\phi_j\}_1^\infty$ of $\mathfrak{H}$ and an operator $T \in \mathfrak{S}_2$ which satisfy the following two conditions*:

1) $$T\phi_j = \psi_j - \phi_j \qquad (j = 1, 2, \cdots),$$

2) *the operator $I + T$ is invertible.*

In fact, if $\{\psi_j\}_1^\infty$ is a basis quadratically close to an orthonormal basis $\{\phi_j\}_1^\infty$, then there exists a bounded invertible operator A which carries the basis $\{\phi_j\}$ into $\{\psi_j\}$. For the operator $T = A - I$ one has

$$T\phi_j = \psi_j - \phi_j \qquad (j = 1, 2, \cdots).$$

Consequently

$$\sum_j |T\phi_j|^2 = \sum_j |\psi_j - \phi_j|^2 < \infty, \tag{3.1}$$

i.e., $T \in \mathfrak{S}_2$.

Conversely, it follows from the conditions 1) and 2) that $\{\psi_j\}_1^\infty$ is a basis of the space $\mathfrak{H}$, and from the condition $T \in \mathfrak{S}_2$ it follows that the basis $\{\psi_j\}$ is quadratically close to the basis $\{\phi_j\}$.

As a corollary of result 2 we obtain:

3. *If the basis $\{\psi_j\}_1^\infty$ is quadratically close to an orthonormal basis $\{\phi_j\}_1^\infty$, then the biorthogonal basis $\{\chi_j\}_1^\infty$ is quadratically close to the basis $\{\phi_j\}_1^\infty$, and consequently the bases $\{\psi_j\}_1^\infty$ and $\{\chi_j\}_1^\infty$ are quadratically close.*

Indeed, let $I + T$ $(T \in \mathfrak{S}_2)$ be an invertible operator which carries the basis $\{\phi_j\}_1^\infty$ into $\{\psi_j\}_1^\infty$. Then from the relations

$$(\phi_k, (I + T^*)\chi_j) = ((I + T)\phi_k, \chi_j) = (\psi_k, \chi_j) = \delta_{jk} \qquad (j, k = 1, 2, \cdots)$$

it follows that

$$\phi_j = (I + T^*)\chi_j \qquad (j = 1, 2, \cdots).$$

Since the operator $T_1 = (I + T^*)^{-1} - I$ belongs to $\mathfrak{S}_2$, the equalities

$$(I + T_1)\phi_j = \chi_j \qquad (j = 1, 2, \cdots)$$

imply the quadratic closeness of the bases $\{\chi_j\}_1^\infty$ and $\{\phi_j\}_1^\infty$. Since

$$|\psi_j - \chi_j|^2 \leqq 2(|\psi_j - \phi_j|^2 + |\chi_j - \phi_j|^2) \qquad (j = 1, 2, \cdots),$$

result 3 is proved.

2. Let $\{\psi_j\}_1^\infty$ be any sequence of linearly independent vectors of the space $\mathfrak{H}$. Then all the Gramians $D(\psi_1, \psi_2, \cdots, \psi_n) = \det \| (\psi_j, \psi_k) \|_1^n$ $(n = 1, 2, \cdots)$ are positive. On the basis of a well-known inequality of Hadamard (see F. R. Gantmaher [**1**], Chapter IX, §5),

$$D(\psi_1, \psi_2, \cdots, \psi_{n+m}) \leqq D(\psi_1, \psi_2, \cdots, \psi_n)\, D(\psi_{n+1}, \cdots, \psi_{n+m}),$$

choosing $m = 1$, we conclude that for the case in which the sequence $\{\psi_j\}_1^\infty$ is normalized $(|\psi_j| = 1)$, the sequence of positive numbers $D(\psi_1, \psi_2, \cdots, \psi_n)$ is nonincreasing, and thus there always exists the limit

$$\Delta = \lim_{n \to \infty} D(\psi_1, \psi_2, \cdots, \psi_n) \geqq 0.$$

It is easily deduced that this limit Δ does not change under any permutation of the unit vectors ψ_j $(j = 1, 2, \cdots)$.

The limit Δ can be regarded as the square of the volume of the parallelepiped spanned by the infinite number of unit vectors ψ_j $(j = 1, 2, \cdots)$.

THEOREM 3.1. *In order that a sequence of unit vectors $\{\psi_j\}_1^\infty$ which is complete in $\mathfrak{H}$ form a basis quadratically close to an orthonormal basis, it is necessary and sufficient that the volume of the parallelepiped determined by them be positive, i.e., that*

$$\Delta = \lim_{n \to \infty} D(\psi_1, \psi_2, \cdots, \psi_n) > 0. \tag{3.2}$$

PROOF. Let $\{\psi_j\}_1^\infty$ be a normalized basis quadratically close to an orthonormal basis. We denote by $\{\chi_j\}_1^\infty$ the system biorthogonal to $\{\psi_j\}_1^\infty$.

As was proved, $\{\chi_j\}_1^\infty$ is a basis quadratically close to an orthonormal basis, and

$$\sum_{j=1}^\infty |\psi_j - \chi_j|^2 < \infty. \tag{3.3}$$

The unit vector $e_j = \chi_j / |\chi_j|$ is orthogonal to the closed linear hull $\mathfrak{M}_j$ of all the ψ_k with $k \neq j$; consequently the distance δ_j from the unit vector ψ_j to $\mathfrak{M}_j$ $(j = 1, 2, \cdots)$ can be calculated from the formula

$$\delta_j = (\psi_j, e_j) = |\chi_j|^{-1} \qquad (0 < \delta_j \leqq 1;\ j = 1, 2, \cdots).$$

Since $|\psi_j - \chi_j|^2 = |\chi_j|^2 - 1$ $(j = 1, 2, \cdots)$, it follows from (3.3) that the series $\sum (\delta_j^{-2} - 1)$ converges, and along with it the series $\sum (1 - \delta_j^2)$ converges. Obviously $\delta_j \leqq d_j$ $(j = 1, 2, \cdots)$, where d_j denotes the distance

from the unit vector ψ_j to the linear hull $\mathfrak{L}_{j-1}$ of the vectors $\psi_1, \psi_2, \dots, \psi_{j-1}$. Thus

$$(3.4) \qquad \sum_{j=1}^{\infty} (1 - d_j^2) < \infty.$$

On the other hand, as is known, $d_j^2 = D_j / D_{j-1}$, where $D_j = D(\psi_1, \psi_2, \cdots, \psi_j)$. Thus

$$\sum_{j=1}^{\infty} \left(1 - \frac{D_j}{D_{j-1}}\right) < \infty,$$

and this inequality is a necessary and sufficient condition for the existence of a positive limit $\lim \prod_1^n (D_j/D_{j-1}) = \lim D_n$ ($D_0 = 1$ by definition). Thus the necessity of condition (3.2) is proved.

We prove its sufficiency. As was just made clear, condition (3.2) is equivalent to condition (3.4), or, since $0 < d_j < 1$, to the condition $\sum_j (1 - d_j) < \infty$. Making use of the process of triangular orthogonalization, we construct an orthonormal basis ϕ_j ($j = 1, 2, \cdots$), where

$$(3.5) \qquad \phi_j = c_{j1}\psi_1 + c_{j2}\psi_2 + \cdots + c_{jj}\psi_j \qquad (j = 1, 2, \cdots),$$

and

$$(3.6) \qquad c_{jj} > 0 \qquad (j = 1, 2, \cdots).$$

Then $\phi_j \in \mathfrak{L}_{j-1}^{\perp}$, $d_j = (\phi_j, \psi_j)$ and

$$(3.7) \qquad |\phi_j - \psi_j|^2 = 2(1 - (\phi_j, \psi_j)) = 2(1 - d_j) \qquad (j = 1, 2, \cdots),$$

so that the condition $\sum_j |\phi_j - \psi_j|^2 < \infty$ will be fulfilled.

It remains to show that the vectors ψ_j ($j = 1, 2, \cdots$) are ω-linearly independent. Let us assume the converse, i.e., that for certain c_j ($j = 1, 2, \cdots$), not all of which equal zero, the condition $\sum_j c_j \psi_j = 0$ is fulfilled. Without loss of generality we may assume that $c_1 = 1$. We will then have

$$\epsilon_n = |\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n|^2 \to 0 \quad \text{for } n \to \infty.$$

On the other hand,

$$\epsilon_n \geqq \min_\xi |\psi_1 + \xi_2\psi_2 + \cdots + \xi_n\psi_n|^2 = D(\psi_1, \psi_2, \cdots, \psi_n)/D(\psi_2, \cdots, \psi_n),$$

and since, by virtue of Hadamard's inequality, $D(\psi_2, \cdots, \psi_n) \leqq |\psi_2| \cdots |\psi_n| = 1$, it follows that $D(\psi_1, \psi_2, \cdots, \psi_n) \leqq \epsilon_n$. We have arrived at a contradiction with condition (3.2). The theorem is proved.

An analysis of the proof of Theorem 3.1 shows that we have at the same time established another theorem:

THEOREM 3.2. *In order that a sequence of unit vectors* $\{\psi_j\}_1^\infty$ *which is complete in* $\mathfrak{H}$ *be a basis of the space* $\mathfrak{H}$ *quadratically close to an orthonormal basis, it is necessary and sufficient that there exist a sequence* $\{\chi_j\}_1^\infty$ *biorthogonal to* $\{\psi_j\}_1^\infty$, *and that these sequences be quadratically close.*

3. For the proof of certain properties of Bari bases we need the following lemma.

LEMMA 3.1. *Let* A *be a bounded linear invertible operator. If* $A^*A - I$ *belongs to* $\mathfrak{S}_2$, *then the operator* $(A^*A)^{1/2} - I$ *also belongs to* $\mathfrak{S}_2$, *and for any unitary operator* U *the inequality*

$$|A - U|_2 \geqq |(A^*A)^{1/2} - I|_2 \tag{3.8}$$

is fulfilled. Equality holds in this relation if and only if U *is the unitary operator from the polar decomposition of the operator* A, *i.e.*

$$U = A(A^*A)^{-1/2}.$$

PROOF. Let $H = (A^*A)^{1/2} - I$; then the polar decomposition of the operator A will have the form $A = U_1(I + H)$, where U_1 is a unitary operator. From the equality

$$H((A^*A)^{1/2} + I) = A^*A - I,$$

by virtue of the fact that the operator $(A^*A)^{1/2} + I$ is invertible, it follows that the operator H belongs to the ideal $\mathfrak{S}_2$.

To prove the relation (3.8), it is sufficient to consider the case of a unitary operator U for which $A - U \in \mathfrak{S}_2$. If we denote by V the unitary operator $U_1^{-1}U$, then

$$|A - U|_2^2 = |I + H - V|_2^2 = \mathrm{sp}[(H + I - V)(H + I - V^*)],$$

and consequently

$$|A - U|_2^2 = \mathrm{sp}\, H^2 + \mathrm{sp}\, C,$$

where

$$C = 2I + 2H - V - V^* - HV^* - VH \tag{3.9}$$

is a selfadjoint operator from $\mathfrak{S}_1$.

Let $\{\chi_j\}$ be a complete orthonormal system of eigenvectors of the operator H:

$$H\chi_j = \lambda_j(H)\chi_j, \quad (\chi_j, \chi_k) = \delta_{jk} \qquad (j, k = 1, 2, \cdots).$$

Then

$$\operatorname{sp} C = \sum_{j=1}^{\infty} (C\chi_j, \chi_j).$$

It is easily seen that by (3.9)

$$(C\chi_j, \chi_j) = 2(\lambda_j(H) + 1)\,[1 - \operatorname{Re}(V\chi_j, \chi_j)] \qquad (j = 1, 2, \cdots).$$

Since

$$1 - \operatorname{Re}(V\chi_j, \chi_j) \geqq 0 \qquad (j = 1, 2, \cdots), \tag{3.10}$$

we have $\operatorname{sp} C \geqq 0$, and so the relation

$$|A - U|_2^2 \geqq \operatorname{sp} H^2,$$

which coincides with (3.8), holds.

The equal sign in (3.8) holds if and only if it holds for all the relations (3.10), i.e.,

$$\operatorname{Re}(V\chi_j, \chi_j) = 1 \qquad (j = 1, 2, \cdots).$$

Bearing in mind that $|(V\chi_j, \chi_j)| \leqq 1$, we obtain $(V\chi_j, \chi_j) = 1$ $(j = 1, 2, \cdots)$. It follows that

$$V\chi_j = \chi_j \qquad (j = 1, 2, \cdots),$$

i.e. $V = I$, or, what is the same, $U = U_1$. The lemma is proved.

THEOREM 3.3 *In order that a sequence $\{\psi_j\}_1^\infty$ which is complete in $\mathfrak{H}$ be a basis quadratically close to an orthonormal basis, it is necessary and sufficient that*

1) *the sequence $\{\psi_j\}_1^\infty$ be ω-linearly independent, and*
2) *the matrix*

$$\| (\psi_j, \psi_k) - \delta_{jk} \|_1^\infty \tag{3.11}$$

be of Hilbert-Schmidt class, i.e.

$$\sum_{j,k=1}^{\infty} |(\psi_j, \psi_k) - \delta_{jk}|^2 < \infty. \tag{3.12}$$

PROOF. The necessity of the first condition is trivial. To prove the necessity of the second condition, we consider a bounded linear invertible operator A, having the property $T = A - I \in \mathfrak{S}_2$, which carries some orthonormal basis $\{\phi_j\}_1^\infty$ into the Bari basis $\{\psi_j\}_1^\infty$. Then

$$(\psi_j, \psi_k) = (A\phi_j, A\phi_k) = (A^*A\phi_j, \phi_k) \qquad (j, k = 1, 2, \cdots).$$

Consequently

$$(\psi_j, \psi_k) - \delta_{jk} = (B\phi_j, \phi_k) \qquad (j, k = 1, 2, \cdots),$$

where $B = T + T^* + T^*T \in \mathfrak{S}_2$, and thus

$$\sum_{j,k=1}^{\infty} |(\psi_j, \psi_k) - \delta_{jk}|^2 = \sum_{j=1}^{\infty} |B\phi_j|^2 = \mathrm{sp}(B^*B) < \infty,$$

i.e. the matrix (3.11) is of Hilbert-Schmidt class.

Let us prove the sufficiency of the hypotheses of the theorem. Starting from any orthonormal basis $\{\phi_j\}_1^\infty$ of the space $\mathfrak{H}$, we form the selfadjoint operator G with matrix (3.11), i.e.

$$((I + G)\phi_j, \phi_k) = (\psi_j, \psi_k) \qquad (j, k = 1, 2, \cdots).$$

It follows from the condition (3.12) that $G \in \mathfrak{S}_2$.

We define an operator A on the linear hull of the vectors $\{\phi_j\}_1^\infty$, putting

$$A\left(\sum_{j=1}^{n} c_j \phi_j\right) = \sum_{j=1}^{n} c_j \psi_j,$$

where the c_j ($j = 1, 2, \cdots, n$; $n = 1, 2, \cdots$) are arbitrary complex numbers. Then it is obvious that

$$|Af|^2 = \sum_{j,k=1}^{n} (\psi_j, \psi_k) c_j \bar{c}_k = ((I + G)f, f) \leqq (1 + |G|)|f|^2$$

for any $f = \sum_{j=1}^{n} c_j \phi_j$.

Thus the operator A can be extended by continuity to the entire space $\mathfrak{H}$. Denoting this extension by the same letter A, we can write

$$|Af|^2 = ((I + G)f, f) \qquad (f \in \mathfrak{H}). \tag{3.13}$$

The operator $I + G$ vanishes only at zero. In fact, if

$$(I + G)\left(\sum_j c_j \phi_j\right) = 0 \quad \left(\sum_j |c_j|^2 < \infty\right),$$

then by (3.13)

$$A\left(\sum_j c_j \phi_j\right) = \sum_j c_j \psi_j = 0,$$

and so $c_1 = c_2 = \cdots = 0$.

From this and the complete continuity of the operator G it follows that the operator $I + G$ is invertible. Consequently there exists a constant $\delta > 0$ such that

$$\delta |f|^2 \leqq ((I + G)f, f) \qquad (f \in \mathfrak{H}).$$

Thus

$$\delta|f|^2 \leqq |Af|^2 \qquad (f \in \mathfrak{H}).$$

In view of the denseness in $\mathfrak{H}$ of the range of the operator A, this last relation implies that A is invertible.

By (3.13), $A^*A - I = G$ $(\in \mathfrak{S}_2)$; consequently, according to Lemma 3.1, $(A^*A)^{1/2} - I \in \mathfrak{S}_2$. Thus

$$|A - U|_2^2 = |(A^*A)^{1/2} - I|_2^2 < \infty,$$

where U is the unitary operator defined by $U = A(A^*A)^{-1/2}$.

We form an orthonormal basis $\{\omega_j\}_1^\infty$ of the space $\mathfrak{H}$, putting $\omega_j = U\phi_j$ $(j = 1, 2, \cdots)$; then

$$\sum_{j=1}^\infty |\psi_j - \omega_j|^2 = \sum_{j=1}^\infty |(A - U)\phi_j|^2 = |A - U|_2^2 < \infty.$$

The theorem is proved.

4. Let us remark that for the orthonormal basis $\{\omega_j\}$, constructed in the proof of the preceding theorem, one has the relation

$$\sum_{j=1}^\infty |\psi_j - \omega_j|^2 = \sum_{j=1}^\infty (\sqrt{1 + \lambda_j} - 1)^2,$$

where $\lambda_j = \lambda_j(G)$ $(j = 1, 2, \cdots)$.

It turns out that the basis $\{\omega_j\}_1^\infty$ is, among all orthonormal bases, *maximally close* to the basis $\{\psi_j\}_1^\infty$, i.e., we have the following theorem.

THEOREM 3.4. *If $\{\psi_j\}_1^\infty$ is a basis which is quadratically close to an orthonormal basis, and $\{\phi_j\}_1^\infty$ is an orthonormal basis, then*

$$\sum_{j=1}^\infty |\psi_j - \phi_j|^2 \geqq \sum_{j=1}^\infty (\sqrt{1 + \lambda_j} - 1)^2, \tag{3.14}$$

where $\{\lambda_j\}$ is the complete system of eigenvalues of the operator which is generated in the space l_2 by the matrix

$$\| (\psi_j, \psi_k) - \delta_{jk} \|_1^\infty.$$

The equal sign in (3.14) *holds for one and only one orthonormal basis, characterized by the condition that all of the matrices $\| (\phi_j, \psi_k) \|_1^n$ $(n = 1, 2, \cdots)$ are positive definite:*[6]

[6] In the paper by M. G. Kreĭn [7], instead of the condition (3.15) there is incorrectly written $(\phi_j, \psi_k) = (\psi_j, \phi_k)$ $(j, k = 1, 2, \cdots)$.

$$\| (\phi_j, \psi_k) \|_1^n \geqq 0. \tag{3.15}$$

PROOF. Let $\{\chi_j\}_1^\infty$ be an arbitrary orthonormal basis and let A be the bounded invertible operator which carries the basis $\{\chi_j\}$ into the basis $\{\psi_j\}$:

$$A\chi_j = \psi_j \qquad (j = 1, 2, \cdots). \tag{3.16}$$

Then

$$(A^*A\chi_j, \chi_k) = (\psi_j, \psi_k) \qquad (j, k = 1, 2, \cdots),$$

so that to the operator A^*A there corresponds, with respect to the basis $\{\chi_j\}$, the matrix $\| (\psi_j, \psi_k) \|_1^\infty$. Therefore, according to Theorem 3.3, $A^*A - I \in \mathfrak{S}_2$. Obviously $\lambda_j(A^*A - I) = \lambda_j$ $(j = 1, 2, \cdots)$.

Let us associate, with every orthonormal basis $\{\phi_j\}_1^\infty$ of the space $\mathfrak{H}$, the unitary operator U which carries the basis $\{\chi_j\}_1^\infty$ into the basis $\{\phi_j\}_1^\infty$. Then

$$\psi_j - \phi_j = (A - U)\chi_j,$$

and so

$$\sum_{j=1}^\infty |\psi_j - \phi_j|^2 = \sum_{j=1}^\infty |(A - U)\chi_j|^2 = |A - U|_2^2.$$

Applying Lemma 3.1 to the operators A and U, we obtain

$$\sum_{j=1}^\infty |\psi_j - \phi_j|^2 \geqq |(A^*A)^{1/2} - I|_2^2 \left(= \sum_{j=1}^\infty (\sqrt{1 + \lambda_j} - 1)^2 \right). \tag{3.17}$$

By the same Lemma 3.1, the equal sign in (3.17) will hold if and only if

$$\phi_j = \omega_j = U\chi_j \qquad (j = 1, 2, \cdots),$$

where U is the unitary operator defined by $U = A(A^*A)^{-1/2}$. Since $\omega_j = U\chi_j$ $(j = 1, 2, \cdots)$, where $U = A(A^*A)^{1/2}$ $(= AS^{-1})$, and A is the bounded linear invertible operator defined by (3.16), we have

$$\left(\sum_{j=1}^n c_j\omega_j, \sum_{j=1}^n c_j\psi_j \right) = \left(\sum_{j=1}^n c_j\chi_j, \sum_{j=1}^n c_jU^{-1}\psi_j \right) = \left(\sum_{j=1}^n c_j\chi_j, \sum_{j=1}^n c_jSA^{-1}\psi_j \right),$$

and so

$$\left(\sum_{j=1}^n c_j\omega_j, \sum_{j=1}^n c_j\psi_j \right) = \left(\sum_{j=1}^n c_j\chi_j, S\left(\sum_{j=1}^n c_j\chi_j \right) \right).$$

Thus the relation (3.15) expresses the fact that the operator S is nonnegative.

To complete the proof of the theorem, it remains to show that an orthonormal basis $\{\phi_j\}_1^\infty$ which satisfies the condition (3.15) is maximally quadratically close to the basis $\{\psi_j\}$. Let us consider the bounded linear invertible operator A which carries the orthonormal basis $\{\phi_j\}$ into the basis $\{\psi_j\}$. Then the condition (3.15) indicates that the operator A is positive definite. Since

$$(A^2\phi_j, \phi_k) = (A\phi_j, A\phi_k) = (\psi_j, \psi_k) \qquad (j, k = 1, 2, \cdots),$$

there corresponds to the operator A^2, with respect to the basis $\{\phi_j\}$, the matrix $\|(\psi_j, \psi_k)\|_1^\infty$. It follows that $A - I \in \mathfrak{S}_2$ and

$$\lambda_j(A - I) = \sqrt{\lambda_j + 1} - 1 \qquad (j = 1, 2, \cdots),$$

where $\{\lambda_j\}$ is the complete system of eigenvalues of the operator which is generated in the space l_2 by the matrix $\|(\psi_j, \psi_k) - \delta_{jk}\|_1^\infty$. Obviously

$$\sum_{j=1}^\infty |\psi_j - \omega_j|^2 = \sum_{j=1}^\infty |(A - I)\omega_j|^2 = \sum_{j=1}^\infty (\sqrt{\lambda_j + 1} - 1)^2.$$

The theorem is proved.

§4. Tests for the existence of a basis, consisting of eigenvectors of a dissipative operator

1. The following result is due to I. M. Glazman [1].

Let $\{\psi_j\}_1^\infty$ be a system of eigenvectors, corresponding to distinct eigenvalues λ_j $(j = 1, 2, \cdots)$ of a dissipative operator. Then the system $\{\psi_j\}$ forms a basis of its closed linear hull, equivalent to an orthonormal basis,[7] *whenever*

$$\sum_{\substack{j,k=1 \\ j \neq k}}^\infty \frac{\operatorname{Im}\lambda_j \operatorname{Im}\lambda_k}{|\lambda_j - \bar{\lambda}_k|} < \infty. \tag{4.1}$$

The first result of this type, but under more restrictive conditions, was obtained by B. R. Mukminov [1]. His proof was based on the triangular model of a dissipative operator, constructed by M. S. Livšic, and was rather complicated. Glazman's proof is elementary and starts from a simple idea, further development of which led to the results presented in this section. To obtain these results an auxiliary result is needed.

2. Let $\{\lambda_j\}$ be the system of all distinct nonreal eigenvalues of an operator $A \in \mathfrak{R}$ with completely continuous imaginary component, and

[7] In essence Glazman's proof contains all the elements necessary for a stronger assertion: under the condition (4.1), the system $\{\psi_j\}$ forms a basis of its closed linear hull, quadratically close to an orthonormal basis.

let $\mathfrak{L}_j$ be the corresponding root subspaces. According to Theorem I.5.2 the numbers λ_j are isolated and the subspaces $\mathfrak{L}_j$ are finite-dimensional.

We choose a number $r_k (> 0; k = 1, 2, \cdots)$ such that the disc $|\lambda - \lambda_k| \leq r_k$ does not contain any $\lambda_j \neq \lambda_k$. Then, as is known (see §2, Chapter I), the formula

$$P_k = -\frac{1}{2\pi i}\int_{|\lambda - \lambda_k| = r_k} (A - \lambda I)^{-1} d\lambda \qquad (k = 1, 2, \cdots)$$

defines a projector (not necessarily orthogonal) onto the root subspace $\mathfrak{L}_k$;

$$P_k\mathfrak{H} = \mathfrak{L}_k \quad \text{and} \quad P_k\mathfrak{L}_j = 0 \quad \text{for} \quad j \neq k.$$

1. *A sequence* $\{\psi_j\}$, *made up of bases of all the subspaces* $\mathfrak{L}_j$, *is* ω-*linearly independent.*

In fact, if

$$\sum_j c_j\psi_j = 0,$$

then

$$P_k\left(\sum_j c_j\psi_j\right) = \sum_{j=m_k+1}^{m_{k+1}} c_j\psi_j,$$

where the ψ_j $(j = m_k + 1, m_k + 2, \cdots, m_{k+1})$ form a basis of the subspace $\mathfrak{L}_k$, and consequently $c_j = 0$ $(j = 1, 2, \cdots)$.

It follows from result 1 that a sequence made up of bases of all of the eigenspaces of the operator A corresponding to nonreal eigenvalues is a fortiori ω-linearly independent.

3. We shall denote the eigenspace of the operator A, corresponding to its eigenvalue λ, by $\mathfrak{Z}_\lambda$.

THEOREM 4.1 *Let A be a bounded linear dissipative operator, with a completely continuous imaginary component, which has a sequence of eigenvalues $\{\lambda_j\}_1^\infty$ $(\lambda_j \neq \lambda_k$ for $j \neq k)$. If*

$$\sum_{\substack{j,k=1 \\ j\neq k}}^{\infty} \min(n_j, n_k) \frac{\operatorname{Im}\lambda_j \operatorname{Im}\lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < \infty, \text{[8)]} \tag{4.2}$$

where $n_j = \dim \mathfrak{Z}_j$ $(\mathfrak{Z}_j = \mathfrak{Z}_{\lambda_j})$, then a sequence made up from orthonormal bases of the subspaces $\mathfrak{Z}_j$ $(j = 1, 2, \cdots)$ forms a basis in its closed linear hull $\mathfrak{Z}$ which is quadratically close to an orthonormal basis. If the weaker condition

8) All terms in the sum (4.2) for which $\operatorname{Im}\lambda_j \operatorname{Im}\lambda_k = 0$ are taken as equal to zero, independent of whether the quantity $\min(n_j, n_k)$ is finite or infinite.

$$\sum_{\substack{j,k=1 \\ j \neq k}}^{\infty} \frac{\operatorname{Im}\lambda_j \operatorname{Im}\lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < \infty \tag{4.2'}$$

is fulfilled, then this sequence forms a basis in $\mathfrak{Z}$ *which is equivalent to an orthonormal basis.*

PROOF. Let us first of all recall (see §1, Chapter V) that for a dissipative operator every eigenspace $\mathfrak{Z}_j$ corresponding to a real eigenvalue λ_j is orthogonal to all other eigenspaces. Therefore we may assume without loss of generality that all the numbers λ_j $(j = 1, 2, \cdots)$ are nonreal, and consequently that all the eigenspaces are finite-dimensional.

Applying the Cauchy-Bunjakovskiĭ inequality to the nonnegative bilinear form $(A_{\mathscr{I}}\phi, \psi)$, we obtain

$$|(A_{\mathscr{I}}\phi, \psi)|^2 \leqq (A_{\mathscr{I}}\phi, \phi)(A_{\mathscr{I}}\psi, \psi).$$

If ϕ and ψ are unit vectors from the subspaces $\mathfrak{Z}_j$ and $\mathfrak{Z}_k$ respectively, then

$$(A_{\mathscr{I}}\phi, \psi) = (1/2i)\,|(A\phi, \psi) - (\phi, A\psi)| = (1/2i)\,(\lambda_j - \bar{\lambda}_k)\,(\phi, \psi)$$

and

$$(A_{\mathscr{I}}\phi, \phi) = \operatorname{Im}(A\phi, \phi) = \operatorname{Im}\lambda_j; \qquad (A_{\mathscr{I}}\psi, \psi) = \operatorname{Im}\lambda_k.$$

Thus

$$|(\phi, \psi)|^2 \leqq 4 \operatorname{Im}\lambda_j \operatorname{Im}\lambda_k / |\lambda_j - \bar{\lambda}_k|^2 \qquad (= c_{jk}). \tag{4.3}$$

Let us denote by ϕ_r $(r = 1, 2, \cdots, n_j)$ and ψ_q $(q = 1, 2, \cdots, n_k)$ orthonormal bases in the subspaces $\mathfrak{Z}_j$ and $\mathfrak{Z}_k$ respectively. Then for the vectors $\psi = \sum_q (\phi_r, \psi_q)\psi_q$ $(\in \mathfrak{Z}_k)$ and $\phi_r \in \mathfrak{Z}_j$ we will have

$$c_{jk} \geqq |(\psi, \phi_r)|^2 = \sum_{q=1}^{n_k} |(\psi_q, \phi_r)|^2 \qquad (r = 1, 2, \cdots, n_j).$$

Consequently

$$\sum_{r=1}^{n_j} \sum_{q=1}^{n_k} |(\psi_q, \phi_r)|^2 \leqq n_j c_{jk}.$$

Interchanging the roles of the subspaces $\mathfrak{Z}_j$ and $\mathfrak{Z}_k$, we obtain the same inequality with n_j replaced by n_k, whence

$$\sum_{r=1}^{n_j} \sum_{q=1}^{n_k} |(\psi_q, \phi_r)|^2 \leqq \min(n_j, n_k) \frac{4 \operatorname{Im}\lambda_j \operatorname{Im}\lambda_k}{|\lambda_j - \bar{\lambda}_k|^2}. \tag{4.4}$$

Let $\mathscr{A}$ denote the Gram matrix of the sequence of vectors consisting

of the union of orthonormal bases of all the $\mathfrak{Z}_j$. Then it follows from (4.2) and (4.4) that the matrix $\mathscr{V} - I$ is of Hilbert-Schmidt class. By virtue of Theorem 3.2 and result 1, this completes the proof of the first part of the theorem.

To prove the second part, we introduce the matrix

$$\mathscr{A}_{jk} = \| (\psi_p, \phi_q) \|_{\substack{p=1,2,\cdots,n_k \\ q=1,2,\cdots,n_j}} \qquad (j, k = 1, 2, \cdots).$$

This matrix generates some operator A_{jk} which maps the coordinates of the unitary space E_{n_k} into the coordinates of the unitary space E_{n_j}; $n_j = \dim E_{n_j}$; $j = 1, 2, \cdots$.

Obviously the matrix $\mathscr{A}_{jk}$ can also be regarded as the matrix in the bases $\{\phi_q\}$ and $\{\psi_p\}$, respectively, defining the projector P_{jk} which orthogonally projects the subspace $\mathfrak{Z}_j$ onto $\mathfrak{Z}_k$:

$$P_{jk}\left(\sum_\nu \xi_\nu \phi_\nu\right) = \sum_\mu \eta_\mu \psi_\mu,$$

where

$$\eta_\mu = \sum_\nu (\phi_\nu, \psi_\mu) \xi_\nu \qquad (\mu = 1, 2, \cdots, n_k).$$

It follows at once that

$$|\mathscr{A}_{jk}| = \max_{\substack{|\phi| = |\psi| = 1 \\ \psi \in \mathfrak{Z}_j; \phi \in \mathfrak{Z}_k}} |(\psi, \phi)|. \tag{4.5}$$

Let $\tilde{\mathfrak{L}}_2$ be the Hilbert space of sequences of vectors $\tilde{x} = \{x_j\}_1^\infty$ $(x_j \in E_{n_j}; j = 1, 2, \cdots)$ with the norm

$$|\tilde{x}| = \left(\sum_{j=1}^\infty |x_j|^2\right)^{1/2}.$$

The spaces $\tilde{\mathfrak{L}}_2$ and l_2 are equivalent. In fact, with every vector $\tilde{x} = \{x_j\}_1^\infty$ $(\in \tilde{\mathfrak{L}}_2)$, where $x_j = \{x_{j1}, x_{j2}, \cdots, x_{jn_j}\}$, we associate the vector

$$x = \{x_{11}, x_{12}, \cdots, x_{1n_1}, x_{21}, \cdots\} \in l_2.$$

Obviously $|\tilde{x}| = |x|$.

The matrix

$$\tilde{\mathfrak{A}} = \|\mathscr{A}_{jk}\|_{j,k=1,2,\cdots}$$

generates a bounded linear operator $\tilde{\mathscr{A}}$ in the space $\tilde{\mathfrak{L}}_2$ for which, by virtue of (4.3) and (4.5), one has the bound

$$|\tilde{\mathfrak{A}} - \tilde{I}|^2 \leqq \sum_{j,k;j\neq k} |\gamma_{jk}|^2 \leqq \sum_{j,k;j\neq k} c_{jk} \quad (< \infty).$$

It follows that the same bound holds for the operator $\hat{A}$ which is generated in l_2 by the Gram matrix $\mathscr{A}$:

$$|\hat{A} - I|^2 \leqq \sum_{j,k=1;j\neq k}^{\infty} c_{jk}. \tag{4.6}$$

It is obvious that the second assertion of the theorem will be established if we show that for sufficiently large N the union of orthonormal bases of all the $\mathfrak{Z}_j$ with $j > N$ forms a basis in its closed linear hull which is equivalent to an orthonormal basis. Therefore we may assume without loss of generality that

$$\sum_{j,k;j\neq k} c_{jk} = 4 \sum_{j,k;j\neq k} \frac{\operatorname{Im}\lambda_j \operatorname{Im}\lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < 1.$$

On the other hand, if this condition is fulfilled, it follows from (4.6) that the matrix $\mathscr{A}$ generates a bounded linear invertible operator in l_2. The theorem is proved.

COROLLARY 4.1. *Suppose that, starting with some integer N, all root subspaces $\mathfrak{L}_{\lambda_j}$ ($j > N$) of the bounded linear dissipative operator A with $A_{\mathscr{J}} \in \mathfrak{S}_\infty$ consist only of eigenvectors, and suppose that the system of all root vectors of A is complete in $\mathfrak{H}$.*

Then when the condition (4.2) ((4.2′)) *is fulfilled, a sequence consisting of orthonormal bases of all the $\mathfrak{L}_{\lambda_j}$ forms a basis of the space $\mathfrak{H}$ which is quadratically close to an orthonormal basis (equivalent to an orthonormal basis).*

§5. Basis of subspaces

1. A sequence $\{\mathfrak{N}_k\}_1^\infty$ of nonzero subspaces $\mathfrak{N}_k \subset \mathfrak{H}$ is said to be a *basis* (of subspaces) of the space $\mathfrak{H}$, if any vector $x \in \mathfrak{H}$ can be expanded in a unique way in a series of the form

$$x = \sum_{k=1}^{\infty} x_k, \tag{5.1}$$

where $x_k \in \mathfrak{N}_k$ ($k = 1, 2, \cdots$).

If the subspaces $\mathfrak{N}_k$ ($k = 1, 2, \cdots$) are one-dimensional, then they form a basis of the space $\mathfrak{H}$ if and only if unit vectors $\phi_k \in \mathfrak{N}_k$ ($k = 1, 2, \cdots$) form a vector basis of $\mathfrak{H}$.

A substantial portion of the results on vector bases carries over to bases of subspaces.

Similarly to the way in which Banach's theorem (on the continuous dependence of the coefficients in an expansion with respect to a basis upon the element) lies at the basis of the results on vector bases, the starting point of the generalization to bases of subspaces is the following result.

1. *Let* $\{\mathfrak{N}_k\}_1^\infty$ *be a basis of subspaces, and let* N_k *be the operator which associates with the vector* x *its component* x_k *from the expansion* (5.1). *Then* $\{N_k\}_1^\infty$ *forms an orthogonal system of continuous projectors, i.e.* $N_jN_k = \delta_{jk}N_k$, *and*

$$\sup_n \left|\sum_{k=1}^n N_k\right| < \infty. \tag{5.2}$$

Let us consider a new norm $\|x\|$, defined in the space $\mathfrak{H}$ by

$$\|x\| = \sup_n \left|\sum_{k=1}^n x_k\right| \qquad (x \in \mathfrak{H}).$$

Just as for the case of vector bases (see Banach [1] or Ljusternik and Sobolev [1]), one can verify without difficulty that the space $\mathfrak{H}$ is complete in the new norm. Since

$$|x| \leqq \|x\| \qquad (x \in \mathfrak{H}), \tag{5.3}$$

according to a well-known theorem of Banach [1] (Chapter VII §3), there exists a constant $c > 0$ such that

$$\|x\| \leqq c|x|. \tag{5.4}$$

It follows in particular that

$$|x_j| \leqq \left|\sum_{k=1}^j x_k\right| + \left|\sum_{k=1}^{j-1} x_k\right| \leqq 2c|x|. \tag{5.5}$$

Let us consider the operator N_k $(k = 1, 2, \cdots)$, which associates with every vector $x \in \mathfrak{H}$ its component x_k from the expansion (5.1). This linear operator is defined on all of the space $\mathfrak{H}$, is bounded by virtue of (5.5), and $N_k^2 = N_k$. Consequently N_k is a bounded projector which projects all of $\mathfrak{H}$ onto $\mathfrak{N}_k$. The expansion (5.1) is equivalent to the expansion

$$x = \sum_{k=1}^\infty N_k x,$$

i.e., the series $\sum_k N_k$ converges strongly to the unit operator. We further note that the relation (5.2) follows from (5.4).

2. It is obvious that every system of pairwise orthogonal subspaces

$\{\mathfrak{N}_k\}_1^\infty$ which is complete[9] in $\mathfrak{H}$ is a basis of $\mathfrak{H}$; in this case the corresponding projectors N_k are orthoprojectors. We shall call such a basis an *orthogonal* basis.

It is easy to show that every bounded invertible operator A transforms any orthogonal basis of subspaces $\{\mathfrak{M}_k\}_1^\infty$ into some other basis $\{\mathfrak{N}_k\}_1^\infty$ of the space $\mathfrak{H}$. A basis of subspaces $\{\mathfrak{N}_k\}$ which is obtained from an orthogonal basis by means of such a transformation is said to be *equivalent to an orthogonal basis.* [10]

A characteristic property (Theorem 2.2) of bases which are equivalent to orthonormal bases is preserved for bases of subspaces which are equivalent to orthogonal ones. To prove this we need the following auxiliary result.

LEMMA 5.1 (G. W. MACKEY [**1**]).[11] *Let $N_1, N_2, \cdots, N_n$ be pairwise orthogonal projectors in $\mathfrak{H}$:*

$$N_j N_k = \delta_{jk} N_j \qquad (j, k = 1, 2, \cdots, n).$$

If

$$\sup_{\epsilon_k = \pm 1} \left| \sum_{k=1}^n \epsilon_k N_k \right| = c, \tag{5.6}$$

then for any $f \in \mathfrak{H}$

$$c^{-2} \left| \sum_{k=1}^n N_k f \right|^2 \leqq \sum_{k=1}^n |N_k f|^2 \leqq c^2 \left| \sum_{k=1}^n N_k f \right|^2.$$

PROOF. By direct verification one can see that

$$\sum_{k=1}^n |N_k f|^2 = \frac{1}{2^n} \sum |\epsilon_1 N_1 f + \cdots + \epsilon_n N_n f|^2 \qquad (f \in \mathfrak{H}),$$

where the summation on the right side extends over all systems $\{\epsilon_k\}_1^n$, where $\epsilon_k = \pm 1$ $(k = 1, 2, \cdots, n)$. From this relation there follows the existence, for every $f \in \mathfrak{H}$, of systems $\{\epsilon_k'\}_1^n$ and $\{\epsilon_k''\}_1^n$ $(\epsilon_k' = \pm 1;\ \epsilon_k'' = \pm 1;$ $k = 1, 2, \cdots, n)$ such that

$$\left| \sum_{k=1}^n \epsilon_k' N_k f \right|^2 \leqq \sum_{k=1}^n |N_k f|^2 \leqq \left| \sum_{k=1}^n \epsilon_k'' N_k f \right|^2. \tag{5.7}$$

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[9] Completeness means that the closed linear hull of these subspaces coincides with $\mathfrak{H}$.

[10] Bases which are equivalent to an orthogonal basis were considered by M. K. Fage [**1**] (he called them *rectifiable*).

[11] This lemma is also given in a paper by J. Wermer [**2**].

Consequently, if we denote the sum $\sum_k N_k$ by N, then on the basis of (5.7) and (5.6) we will have

$$\sum_{k=1}^{n} |N_k f|^2 \leqq \left| \sum_{k=1}^{n} \epsilon_k'' N_k N f \right|^2 \leqq c^2 |Nf|^2.$$

Since

$$\left(\sum_{k=1}^{n} \epsilon_k' N_k \right)^2 = N,$$

we have

$$|Nf|^2 = \left| \left(\sum_{k=1}^{n} \epsilon_k' N_k \right)^2 f \right|^2 \leqq \left| \sum_{k=1}^{n} \epsilon_k' N_k \right|^2 \left| \sum_{k=1}^{n} \epsilon_k' N_k f \right|^2.$$

It follows, by virtue of (5.6) and (5.7), that

$$|Nf|^2 \leqq c^2 \sum_{k=1}^{n} |N_k f|^2 \qquad (f \in \mathfrak{H}).$$

The lemma is proved.

THEOREM 5.1 (I. M. GEL'FAND [1]). *In order that a sequence $\{\mathfrak{N}_j\}$ of subspaces be a basis of the space $\mathfrak{H}$ which is equivalent to an orthogonal one, it is necessary and sufficient that for any permutation of its elements this sequence remain a basis of $\mathfrak{H}$.*

PROOF. The necessity of the hypothesis of the theorem is obvious. We prove its sufficiency. Let N_k $(k = 1, 2, \cdots)$ be the projector which projects the space $\mathfrak{H}$ onto $\mathfrak{N}_k$ parallel to the closed linear hull of all the subspaces $\mathfrak{N}_j$ $(j \neq k)$. If $\{N_{k'}\}$ is any permutation of the sequence $\{N_k\}$, then by the hypothesis of the theorem and inequality (5.2) the partial sums of the series $\sum N_{k'}$ are bounded. Consequently, according to Lemma 2.1,

$$\sup_{n; \epsilon_k = \pm 1} \left| \sum_{k=1}^{n} \epsilon_k N_k \right| = c < \infty.$$

Thus by Lemma 5.1, for any vector $f \in \mathfrak{H}$ one has

$$c^{-2} \left| \sum_{k=1}^{n} N_k f \right|^2 \leqq \sum_{k=1}^{n} |N_k f|^2 \leqq c^2 \left| \sum_{k=1}^{n} N_k f \right|^2 \qquad (n = 1, 2, \cdots). \tag{5.8}$$

Let us choose an orthogonal basis $\{\mathfrak{M}_j\}$ in the space $\mathfrak{H}$, subject to the condition

$$\dim \mathfrak{M}_j = \dim \mathfrak{N}_j \qquad (j = 1, 2, \cdots),$$

and denote by A_j an isometric mapping of $\mathfrak{M}_j$ onto $\mathfrak{N}_j$. We define a linear operator A on the linear hull of all the subspaces $\mathfrak{M}_j$, putting

$$A\phi = \sum_{k=1}^{\infty} A_k P_k \phi \qquad (\phi \in \mathfrak{H}),$$

where P_k is the orthoprojector which projects $\mathfrak{H}$ onto the subspace $\mathfrak{M}_k$ $(k = 1, 2, \cdots)$. Setting $f = A\phi$ in (5.8), we obtain

$$c^{-1}|A\phi| \leqq |\phi| \leqq c|A\phi|.$$

Thus the operator A can be extended by continuity to a bounded invertible operator. Obviously $A\mathfrak{M}_k = \mathfrak{N}_k$, and consequently $\{\mathfrak{N}_k\}_1^\infty$ is a basis equivalent to an orthogonal one. The theorem is proved.

3. We need a number of definitions.

A sequence $\{\mathfrak{N}_k\}_1^\infty$ of nonzero subspaces is said to be *ω-linearly independent* if the equality

$$\sum_{k=1}^{\infty} x_k = 0 \qquad (x_k \in \mathfrak{N}_k;\, k = 1, 2, \cdots)$$

is not possible for

$$0 < \sum_{k=1}^{\infty} |x_k|^2 < \infty.$$

If P and Q are the orthoprojectors onto subspaces $\mathfrak{M}$ and $\mathfrak{N}$ respectively, then, as we know (see §3, Chapter I), the number $|P - Q|$ is in a certain sense a measure of the deviation of the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ from one another. Therefore the following definition is natural.

Two sequences of subspaces $\{\mathfrak{M}_k\}_1^\infty$ and $\{\mathfrak{N}_k\}_1^\infty$ are said to be *quadratically close* if

$$\sum_{k=1}^{\infty} |P_k - Q_k|^2 < \infty,$$

where P_k and Q_k are the orthoprojectors onto the subspaces $\mathfrak{M}_k$ and $\mathfrak{N}_k$ respectively.

Let us remark that two sequences $\{\mathfrak{M}_k\}_1^\infty$ and $\{\mathfrak{N}_k\}_1^\infty$ of one-dimensional subspaces are quadratically close if and only if one can choose unit vectors $x_k \in \mathfrak{M}_k$ and $y_k \in \mathfrak{N}_k$ such that the sequences $\{x_k\}_1^\infty$ and $\{y_k\}_1^\infty$ are quadratically close in the sense of the definition in §2.2, i.e. $\sum |x_k - y_k|^2 < \infty$.

In fact, for the case being considered[12]

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[12] The relation (5.9) can be obtained directly without difficulty; it can be obtained in a completely simple way on the basis of result 4 (p. 340).

(5.9) $$|P_k - Q_k|^2 = 1 - |(x_k, y_k)|^2.$$

Since

(5.10) $$|x_k - y_k|^2 = 2(1 - \operatorname{Re}(x_k, y_k)),$$

and for α with $|\alpha| \leqq 1$

$$1 - |\alpha|^2 \leqq 2(1 - |\alpha|) \leqq 2(1 - \operatorname{Re}\alpha),$$

we have

$$|P_k - Q_k|^2 \leqq |x_k - y_k|^2.$$

Therefore from the quadratic closeness of the sequences $\{x_k\}_1^\infty$ and $\{y_k\}_1^\infty$ (for some choice of unit vectors $x_k \in \mathfrak{M}_k$ and $y_k \in \mathfrak{N}_k$) follows the quadratic closeness of the sequences $\{\mathfrak{M}_k\}_1^\infty$ and $\{\mathfrak{N}_k\}_1^\infty$.

If, conversely, the sequences $\{\mathfrak{M}_k\}_1^\infty$ and $\{\mathfrak{N}_k\}_1^\infty$ are quadratically close, then, choosing unit vectors $x_k \in \mathfrak{M}_k$ and $y_k \in \mathfrak{N}_k$ such that

$$(x_k, y_k) > 0 \qquad (k = 1, 2, \cdots),$$

by virtue of (5.9) and (5.10) we obtain

$$|x_k - y_k|^2 = 2(1 - (x_k, y_k)) \leqq 2(1 - (x_k, y_k)^2) = 2|P_k - Q_k|.$$

Thus the sequences $\{x_k\}_1^\infty$ and $\{y_k\}_1^\infty$ are likewise quadratically close.

Henceforth we shall for simplicity restrict ourselves to the consideration of finite-dimensional subspaces (cf. Remark 5.3).

The following theorem is the analog of Theorem 2.3.

THEOREM 5.2 (A. S. MARKUS [2]). *An ω-linearly independent sequence $\{\mathfrak{N}_k\}_1^\infty$ of finite-dimensional subspaces which is complete in $\mathfrak{H}$ and quadratically close to some basis $\{\mathfrak{M}_k\}_1^\infty$ of the space $\mathfrak{H}$ which is equivalent to an orthogonal basis, is likewise a basis of the space $\mathfrak{H}$ equivalent to an orthogonal basis.*

PROOF. Let $\{\mathfrak{Q}_k\}_1^\infty$ be an orthogonal basis of the space $\mathfrak{H}$ which is transformed by the bounded linear invertible operator A into the basis $\{\mathfrak{M}_k\}_1^\infty$. We choose a number n sufficiently large that

$$\sum_{k=n}^{\infty} |P_k - Q_k|^2 < (|A|\,|A^{-1}|)^{-2},$$

and put

$$B = \sum_{k=n}^{\infty} (Q_k - P_k)\, A R_k,$$

where P_k, Q_k and R_k are the orthoprojectors which project $\mathfrak{H}$ onto the

subspaces $\mathfrak{M}_k$, $\mathfrak{N}_k$ and $\mathfrak{Q}_k$ respectively. Obviously

$$|Bx|^2 \leqq \left(\sum_{k=n}^{\infty} |Q_k - P_k|\,|A|\,|R_k x|\right)^2$$
$$\leqq |A|^2 \sum_{k=n}^{\infty} |Q_k - P_k|^2 \sum_{k=n}^{\infty} |R_k x|^2$$
$$\leqq |A|^2 \sum_{k=n}^{\infty} |Q_k - P_k|^2 |x|^2.$$

Thus

$$|B| < |A^{-1}|^{-1},$$

and therefore the operator $A + B$ is invertible. It is obvious that for $x \in \mathfrak{Q}_k$ $(k \geqq n)$

$$(A + B)x = Q_k A x.$$

Since

$$|P_k - Q_k| < 1 \qquad (k \geqq n),$$

the projector Q_k maps the subspace $\mathfrak{M}_k$ isomorphically onto $\mathfrak{N}_k$ (see Lemma I.3.1). Consequently the operator $A + B$ maps the subspace $\mathfrak{Q}_k$ isomorphically onto the subspace $\mathfrak{N}_k$ $(k \geqq n)$.

Let us denote by $\mathfrak{Q}$ $(\mathfrak{N})$ the direct sum of the subspaces $\mathfrak{Q}_k$ $(k = 1, 2, \cdots, n-1)$ $(\mathfrak{N}_k (k = 1, 2, \cdots, n-1))$ and by $\widetilde{\mathfrak{Q}}$ $(\widetilde{\mathfrak{N}})$ the closed linear hull of the subspaces $\mathfrak{Q}_k$ $(k = n, n+1, \cdots)$ $(\mathfrak{N}_k (k = n, n+1, \cdots))$. Obviously

$$\mathfrak{Q} \oplus \widetilde{\mathfrak{Q}} = \mathfrak{H}. \tag{5.11}$$

We shall show that

$$\mathfrak{N} \dot{+} \widetilde{\mathfrak{N}} = \mathfrak{H}. \tag{5.12}$$

For this it is obviously sufficient to establish that

$$\mathfrak{N} \cap \widetilde{\mathfrak{N}} = 0.$$

Let us consider an arbitrary vector $x \in \mathfrak{N} \cap \widetilde{\mathfrak{N}}$. Since $x \in \mathfrak{N}$, we have

$$x = \sum_{k=1}^{n-1} x_k \qquad (x_k \in \mathfrak{N}_k). \tag{5.13}$$

On the other hand, since $x \in \widetilde{\mathfrak{N}}$ we have $x = (A + B)y, y \in \widetilde{\mathfrak{Q}}$. Representing y in the form

$$y = \sum_{k=n}^{\infty} y_k \qquad (y_k \in \mathfrak{Q}_k),$$

we obtain

$$(5.14) \qquad x = \sum_{k=n}^{\infty} x_k; \qquad x_k = (A+B)\, y_k \in \mathfrak{N}_k.$$

From (5.13) and (5.14) and the ω-linear independence of the sequence $\{\mathfrak{N}_k\}$ it follows that $x_k = 0$ $(k = 1, 2, \cdots)$, and so $x = 0$.

Since $(A+B)\tilde{\mathfrak{Q}} = \tilde{\mathfrak{N}}$, by (5.11) and (5.12) we have $\dim \mathfrak{Q} = \dim \mathfrak{N}$, and therefore the subspace $\mathfrak{Q}$ can be decomposed into the orthogonal sum of subspaces $\hat{\mathfrak{Q}}_k$ $(k = 1, 2, \cdots, n-1)$ such that $\dim \hat{\mathfrak{Q}}_k = \dim \mathfrak{N}_k$ $(k = 1, 2, \cdots, n-1)$. Let $\hat{C}$ be a linear invertible mapping of the subspace $\mathfrak{Q}$ onto the subspace $\mathfrak{N}$ which carries the subspace $\hat{\mathfrak{Q}}_j$ onto the subspace $\mathfrak{N}_j$. We denote by C the linear operator which coincides with $A + B$ on the subspace $\tilde{\mathfrak{Q}}$, and with $\hat{C}$ on the subspace $\mathfrak{Q}$. The bounded invertible operator C transforms the orthogonal basis $\hat{\mathfrak{Q}}_1, \hat{\mathfrak{Q}}_2, \cdots, \hat{\mathfrak{Q}}_{n-1}, \hat{\mathfrak{Q}}_n, \mathfrak{Q}_{n+1}, \cdots$ into the sequence $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$. Consequently $\{\mathfrak{N}_j\}_1^\infty$ is a basis of the space $\mathfrak{H}$ equivalent to an orthogonal basis. The theorem is proved.

REMARK 5.1. In contrast to that which holds for vector bases (Theorem 2.3), the condition of completeness for the sequence $\{\mathfrak{N}_k\}$ in Theorem 5.2 cannot, generally speaking, be discarded. This condition is superfluous if $\dim \mathfrak{N}_k = \dim \mathfrak{M}_k$ $(k = 1, 2, \cdots)$.

4. By the *minimal angle* between two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ is meant the angle $\phi(\mathfrak{M}, \mathfrak{N})$ $(0 \leqq \phi \leqq \pi/2)$, defined by

$$\cos \phi(\mathfrak{M}, \mathfrak{N}) = \sup_{\substack{x \in \mathfrak{M},\, y \in \mathfrak{N} \\ |x| = |y| = 1}} |(x, y)|.$$

We shall present some auxiliary geometric results connected with the concept of the minimal angle.[13] We again denote by $\mathfrak{M}$ and $\mathfrak{N}$ subspaces of the space $\mathfrak{H}$.

2. *If*

$$\mathfrak{M} \dotplus \mathfrak{N} = \mathfrak{H}$$

and M is the projector which projects $\mathfrak{H}$ onto $\mathfrak{M}$ parallel to $\mathfrak{N}$, then

$$\sin \phi(\mathfrak{M}, \mathfrak{N}) = |M|^{-1}.$$

In fact,

[13] Results 2 and 4 are taken from a paper by V. Ė. Ljance [1], and result 3 from a paper by Kreĭn, Krasnosel'skiĭ and Mil'man [1].

$$|M|^{-1} = \inf_{\substack{x \in \mathfrak{M} \\ |x|=1}} \inf_{Mz=x} |z| = \inf_{\substack{x \in \mathfrak{M} \\ |x|=1}} \inf_{y \in \mathfrak{N}} |x-y|$$

$$= \inf_{\substack{x \in \mathfrak{M} \\ |x|=1}} |x - Qx| = \inf_{\substack{x \in \mathfrak{M} \\ |x|=1}} (1 - |Qx|^2)^{1/2},$$

where Q is the orthoprojector onto the subspace $\mathfrak{N}$. Since

$$|Qx| = \sup_{\substack{y \in \mathfrak{N} \\ |y|=1}} |(x,y)|, \tag{5.15}$$

it follows that

$$|M|^{-1} = (1 - \sup_{\substack{x \in \mathfrak{M}, y \in \mathfrak{N} \\ |x|=|y|=1}} |(x,y)|^2)^{1/2} = \sin \phi(\mathfrak{M}, \mathfrak{N}).$$

3. *If*

$$\mathfrak{M} \dotplus \mathfrak{N} = \mathfrak{H}, \ \mathfrak{M}_1 = \mathfrak{M}^{\perp} \quad \textit{and} \quad \mathfrak{N}_1 = \mathfrak{N}^{\perp}, \tag{5.16}$$

then

$$\phi(\mathfrak{M}, \mathfrak{N}) = \phi(\mathfrak{M}_1, \mathfrak{N}_1).$$

We first of all note that by (5.16)

$$\mathfrak{M}_1 \dotplus \mathfrak{N}_1 = \mathfrak{H}.$$

Let us denote by N_1 the projector which projects $\mathfrak{H}$ onto $\mathfrak{N}_1$ parallel to $\mathfrak{M}_1$, and by M the projector from result 2. If x and y are arbitrary vectors from $\mathfrak{H}$, then, putting

$$x = x_1 + x_2 \ (x_1 \in \mathfrak{M}, x_2 \in \mathfrak{N}), \quad y = y_1 + y_2 \ (y_1 \in \mathfrak{M}_1, y_2 \in \mathfrak{N}_1),$$

we obtain

$$(Mx, y) = (x_1, y) = (x_1, y_2) = (x, N_1 y).$$

Thus $N_1 = M^*$ and therefore, by result 2,

$$\sin \phi(\mathfrak{M}, \mathfrak{N}) = |M|^{-1} = |N_1|^{-1} = \sin \phi(\mathfrak{M}_1, \mathfrak{N}_1).$$

4. *If*

$$\mathfrak{M}_1 = \mathfrak{M}^{\perp} \quad \textit{and} \quad \mathfrak{N} \dotplus \mathfrak{M}_1 = \mathfrak{H},$$

then

$$|P - Q| = \cos \phi(\mathfrak{N}, \mathfrak{M}_1), \tag{5.17}$$

where P and Q are the orthoprojectors onto the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

Since

$$((P-Q)x,(P-Q)y) = -((I-Q)x,Qy) = -(Q(I-Q)x,y) = 0$$

for $x \in \mathfrak{M}$ and $y \in \mathfrak{M}_1$, we have

$$|P-Q| = \max(|P-Q|_{\mathfrak{M}}, |P-Q|_{\mathfrak{M}_1}), \tag{5.18}$$

where we have put

$$|A|_{\mathfrak{M}} = \sup_{x\in\mathfrak{M};|x|=1} |Ax|.$$

It is obvious that

$$|P-Q|_{\mathfrak{M}_1} = |Q|_{\mathfrak{M}_1},$$

and since by (5.15)

$$|Q|_{\mathfrak{M}_1} = \sup_{\substack{x\in\mathfrak{M}_1, y\in\mathfrak{N}\\ |x|=|y|=1}} |(x,y)| = \cos\phi(\mathfrak{N},\mathfrak{M}_1),$$

we obtain

$$|P-Q|_{\mathfrak{M}_1} = \cos\phi(\mathfrak{N},\mathfrak{M}_1). \tag{5.19}$$

Replacing in the last equality $\mathfrak{M}_1$ by $\mathfrak{M}$ and $\mathfrak{N}$ by $\mathfrak{N}_1 = \mathfrak{N}^{\perp}$, and, correspondingly, P by $I-P$ and Q by $I-Q$, we obtain

$$|P-Q|_{\mathfrak{M}} = \cos\phi(\mathfrak{N}_1,\mathfrak{M}). \tag{5.20}$$

Since by result 3 the right sides of the relations (5.19) and (5.20) are equal, the relation (5.17) follows from (5.18).

5. It follows from Theorem 5.2, in particular, that every ω-linearly independent sequence of finite-dimensional subspaces $\{\mathfrak{N}_j\}_1^\infty$ which is complete in $\mathfrak{H}$ and quadratically close to some orthogonal basis of the space $\mathfrak{H}$, is a basis of $\mathfrak{H}$.

By Theorem 5.2 every basis which is quadratically close to an orthogonal one is a basis equivalent to an orthogonal one.

The following theorem is a partial generalization of Theorem 3.3.

THEOREM 5.3 (A. S. MARKUS [2]). *Let $\{\mathfrak{N}_k\}_1^\infty$ be an ω-linearly independent sequence of finite-dimensional subspaces which is complete in $\mathfrak{H}$ and such that*

$$\sum_{\substack{j,k=1\\ j\neq k}}^{\infty} \cos^2\phi(\mathfrak{N}_j,\mathfrak{N}_k) < \infty. \tag{5.21}$$

Then $\{\mathfrak{N}_k\}_1^\infty$ is a basis of the space $\mathfrak{H}$ quadratically close to an orthogonal one.

PROOF. We choose a positive integer n such that

$$\sum_{j,k=n;j\neq k}^{\infty} \cos^2\phi(\mathfrak{N}_j, \mathfrak{N}_k) = q^2 < 1 \tag{5.22}$$

and introduce the subspaces

$$\mathfrak{Q}_k = \mathfrak{N}_n \dotplus \mathfrak{N}_{n+1} \dotplus \cdots \dotplus \mathfrak{N}_k \qquad (k = n, n+1, \cdots)$$

and

$$\mathfrak{M}_n = \mathfrak{N}_n, \quad \mathfrak{M}_k = \mathfrak{Q}_k \ominus \mathfrak{Q}_{k-1} \qquad (k = n+1, n+2, \cdots).$$

Let us further denote by $\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{M}_{n-1}$ some $n-1$ subspaces which extend the sequence $\{\mathfrak{M}_k\}_n^\infty$ to an orthogonal basis $\{\mathfrak{M}_k\}_1^\infty$ of the space $\mathfrak{H}$, and by P_k and Q_k the orthoprojectors onto the subspaces $\mathfrak{M}_k$ and $\mathfrak{N}_k$ respectively $(k = 1, 2, \cdots)$.

Since the operator $P_k - Q_k$ equals zero on the subspace $\mathfrak{H} \ominus \mathfrak{Q}_k$ $(k \geq n)$, we have

$$|P_k - Q_k| = |P_k - Q_k|_{\mathfrak{Q}_k}$$

and therefore, by result 4,

$$|P_k - Q_k| = \cos\phi(\mathfrak{N}_k, \mathfrak{Q}_{k-1}) \qquad (k = n+1, n+2, \cdots). \tag{5.23}$$

By Theorem 5.2 the theorem will be proved as soon as it is established that the sequences $\{\mathfrak{M}_k\}_1^\infty$ and $\{\mathfrak{N}_k\}_1^\infty$ are quadratically close. For this, by virtue of (5.23), it suffices to show that

$$\sum_{k=n+1}^{\infty} \cos^2\phi(\mathfrak{N}_k, \mathfrak{Q}_{k-1}) < \infty.$$

Let x and y be arbitrary unit vectors from the subspaces $\mathfrak{N}_k$ and $\mathfrak{Q}_{k-1}$ respectively. We represent the vector y in the form

$$y = \sum_{j=n}^{k-1} \alpha_j x_j \qquad (x_j \in \mathfrak{N}_j, |x_j| = 1; \quad j = n, n+1, \cdots, k-1).$$

Since

$$\begin{aligned} 1 = \left| \sum_{j=n}^{k-1} \alpha_j x_j \right|^2 &= \sum_{p,j=n}^{k-1} (x_p, x_j)\, \alpha_p \overline{\alpha_j} \\ &= \sum_{j=n}^{k-1} |\alpha_j|^2 + \sum_{\substack{p,j=n \\ p\neq j}}^{k-1} (x_p, x_j)\, \alpha_p \overline{\alpha_j} \\ &\geq \sum_{j=n}^{k-1} |\alpha_j|^2 - \left(\sum_{\substack{p,j=n \\ p\neq j}}^{k-1} |(x_p, x_j)|^2 \right)^{1/2} \sum_{j=n}^{k-1} |\alpha_j|^2 \end{aligned}$$

and

$$|(x_p, x_j)| \leqq \cos\phi(\mathfrak{N}_p, \mathfrak{N}_j),$$

it follows from (5.22) that

$$1 \geqq \sum_{j=n}^{k-1} |\alpha_j|^2 - q\sum_{j=n}^{k-1} |\alpha_j|^2,$$

or

$$\sum_{j=n}^{k-1} |\alpha_j|^2 \leqq (1-q)^{-1}.$$

On the other hand,

$$|(x,y)|^2 = \left|\sum_{j=n}^{k-1} (x, x_j)\,\overline{\alpha_j}\right|^2 \leqq \sum_{j=n}^{k-1} |\alpha_j|^2 \sum_{j=n}^{k-1} |(x, x_j)|^2$$
$$\leqq (1-q)^{-1}\sum_{j=n}^{k-1} \cos^2\phi(\mathfrak{N}_k, \mathfrak{N}_j).$$

Thus

$$\cos^2\phi(\mathfrak{N}_k, \mathfrak{L}_{k-1}) \leqq (1-q)^{-1}\sum_{j=n}^{k-1} \cos^2\phi(\mathfrak{N}_k, \mathfrak{N}_j) \qquad (k \geqq n)$$

and therefore

$$\sum_{k=n+1}^{\infty} \cos^2\phi(\mathfrak{N}_k, \mathfrak{L}_{k-1}) \leqq \frac{1}{2}(1-q)^{-1} \sum_{\substack{k,j=n \\ k\neq j}}^{\infty} \cos^2\phi(\mathfrak{N}_k, \mathfrak{N}_j) \leqq \frac{q^2}{2(1-q)}.$$

The theorem is proved.

REMARK 5.2. In contrast with the case of vector bases (Theorem 3.2), the condition (5.21) is not necessary in order that the system $\{\mathfrak{N}_k\}_1^\infty$ form a basis of subspaces quadratically close to an orthogonal one. The necessity of condition (5.21) can be established under the additional restriction

$$\sup_k \dim \mathfrak{N}_k < \infty.$$

REMARK 5.3. In Theorems 5.2 and 5.3 the condition of the finite-dimensionality of the subspaces $\mathfrak{N}_k$ $(k = 1, 2, \cdots)$ can be discarded (the proofs remain unchanged), if the condition of the ω-linear independence of the sequence $\{\mathfrak{N}_k\}_1^\infty$ is replaced by the following stronger condition: for any k the minimal angle between the subspace $\mathfrak{N}_k$ and the closed linear hull of all the other subspaces $\mathfrak{N}_j$ $(j \neq k)$ is positive.

6. We shall present some simple results which establish connections between bases of subspaces and vector bases.

5. *If the sequence of subspaces* $\{\mathfrak{N}_k\}_1^\infty$ *is a basis of the space* $\mathfrak{H}$ *equivalent to an orthogonal one, then any sequence* $\{\phi_j\}_1^\infty$, *obtained as the union of orthonormal bases of all the subspaces* $\mathfrak{N}_k$ $(k = 1, 2, \cdots)$, *is a basis of the space* $\mathfrak{H}$ *equivalent to an orthonormal one.*

In fact, let A be a bounded linear invertible operator which transforms some orthogonal basis $\{\mathfrak{M}_k\}_1^\infty$ of the space $\mathfrak{H}$ into the basis $\{\mathfrak{N}_k\}_1^\infty$. We represent an arbitrary vector $x \in \mathfrak{H}$ in the form

$$x = \sum_{k=1}^{\infty} x_k \qquad (x_k \in \mathfrak{N}_k, k = 1, 2, \cdots).$$

Obviously

$$c^{-1}|x|^2 \leqq \sum_{k=1}^{\infty} |x_k|^2 \leqq c|x|^2 \qquad (c = |A|^2 |A^{-1}|^2) \tag{5.24}$$

Expanding every vector x_k with respect to the given orthonormal basis of the subspace $\mathfrak{N}_k$ $(k = 1, 2, \cdots)$, we obtain an expansion of the vector x with respect to the sequence $\{\phi_j\}_1^\infty$:

$$x = \sum_{j=1}^{\infty} \alpha_j \phi_j,$$

where, by (5.24),

$$c^{-1}|x|^2 \leqq \sum_{j=1}^{\infty} |\alpha_j|^2 \leqq c|x|^2.$$

From the last relation and Theorem 2.1 it follows that $\{\phi_j\}_1^\infty$ is a basis of the space $\mathfrak{H}$ equivalent to an orthonormal one.

If the sequence of subspaces $\{\mathfrak{N}_k\}_1^\infty$ is a basis of the space $\mathfrak{H}$ which is quadratically close to an orthogonal basis, then although, by result 5, any sequence of vectors obtained as the union of orthonormal bases of the subspaces $\mathfrak{N}_k$ $(k = 1, 2, \cdots)$ will be a basis of the space $\mathfrak{H}$ which is equivalent to an orthonormal basis, it will not, generally speaking, be a basis which is quadratically close to an orthonormal one.

We shall give a sufficient condition under which such a sequence of vectors is a basis quadratically close to an orthonormal basis.

6. *If* $\{\mathfrak{N}_k\}_1^\infty$ *is an* ω*-linearly independent sequence of finite-dimensional subspaces which is complete in* $\mathfrak{H}$, *and if*

$$\sum_{j,k=1;\, j\neq k}^{\infty} \min(n_j, n_k) \cos^2\phi(\mathfrak{N}_j, \mathfrak{N}_k) < \infty \qquad (n_j = \dim \mathfrak{N}_j), \tag{5.25}$$

then the union of orthonormal bases $\phi_{k1}, \phi_{k2}, \cdots, \phi_{kn_k}$ *of the subspaces* $\mathfrak{N}_k$ $(k = 1, 2, \cdots)$ *forms a basis of the space* $\mathfrak{H}$ *which is quadratically close to an orthonormal one.*

To prove this result we shall make use of the method applied above in the proof of the first part of Theorem 4.1.

Putting

$$\phi = \sum_{p=1}^{n_j} (\phi_{kq}, \phi_{jp})\, \phi_{jp}\, (\in \mathfrak{N}_j) \qquad (j, k = 1, 2, \cdots; q = 1, 2, \cdots, n_k),$$

we will have

$$\cos^2 \phi(\mathfrak{N}_j, \mathfrak{N}_k) \geqq |(\phi, \phi_{kq})|^2 = \sum_{p=1}^{n_j} |(\phi_{kq}, \phi_{jp})|^2 \qquad (q = 1, 2, \cdots, n_k).$$

Thus

$$\sum_{q=1}^{n_k} \sum_{p=1}^{n_j} |(\phi_{kq}, \phi_{jp})|^2 \leqq n_k \cos^2 \phi(\mathfrak{N}_j, \mathfrak{N}_k).$$

Interchanging the roles of the subspaces $\mathfrak{N}_j$ and $\mathfrak{N}_k$, we obtain the same inequality with n_k replaced by n_j in the right side. Consequently

$$\sum_{q=1}^{n_k} \sum_{p=1}^{n_j} |(\phi_{kq}, \phi_{jp})|^2 \leqq \min(n_j, n_k) \cos^2 \phi(\mathfrak{N}_j, \mathfrak{N}_k). \tag{5.26}$$

Let us number all the vectors ϕ_{kq} $(k = 1, 2, \cdots; q = 1, 2, \cdots, n_k)$ in succession and denote the sequence thus obtained by $\{\phi_j\}_1^\infty$. It follows from (5.25) and (5.26) that

$$\sum_{j,k=1; j \neq k} |(\phi_j, \phi_k)|^2 < \infty.$$

Since the sequence $\{\phi_j\}_1^\infty$ is complete in $\mathfrak{H}$ and ω-linearly independent, by Theorem 3.3 it is a basis of the space $\mathfrak{H}$ which is quadratically close to an orthonormal one.

§6. Tests for the existence of a basis consisting of the root subspaces of a dissipative operator

This section generalizes the results of §4, in that here we consider not only eigenvectors, but also root vectors of the operator, and besides vector bases, also bases consisting of root subspaces.

1. Let ϕ be a root vector of the operator A corresponding to the eigenvalue λ. By the *order* m_ϕ of the root vector ϕ we shall mean the smallest positive integer m such that

$$(A - \lambda I)^m \phi = 0.$$

By the *order* of the eigenvalue λ of A we shall mean the number

$$m_\lambda = \sup_{\phi \in \mathfrak{L}_\lambda} m_\phi,$$

where $\mathfrak{L}_\lambda$ is the root subspace of A corresponding to the eigenvalue λ. Obviously m_λ is the smallest positive integer m for which

$$(A - \lambda I)^m \mathfrak{L}_\lambda = 0.$$

We denote by ν_λ the algebraic multiplicity of the eigenvalue λ of the operator A, i.e.

$$\nu_\lambda = \dim \mathfrak{L}_\lambda.$$

As is known, always $m_\lambda \leqq \nu_\lambda$.

We shall prove two auxiliary results which establish certain bounds for the root vectors of a dissipative operator.

LEMMA 6.1. *Let ϕ be a root vector of the dissipative operator A corresponding to the eigenvalue λ. Then*

$$\operatorname{Im}(A\phi, \phi) \leqq m_\phi \operatorname{Im}\lambda |\phi|^2, \tag{6.1}$$

$$|(A - \lambda I)\phi| \leqq \sqrt{2}\operatorname{Im}\lambda \sqrt{m_\phi(m_\phi - 1)}\,|\phi|. \tag{6.2}$$

PROOF. Let $\mathfrak{N}$ be the linear hull of the vectors

$$\phi_k = (A - \lambda I)^k \phi \qquad (k = 0, 1, \cdots, m-1; m = m_\phi).$$

We denote by $\hat{A}$ the restriction of the operator A to the subspace $\mathfrak{N}$. Obviously $\operatorname{sp}\hat{A} = m\lambda$, and so $\operatorname{sp}\hat{A}_{\mathscr{J}} = m \operatorname{Im}\lambda$. Since the operator $\hat{A}_{\mathscr{J}}$ is nonnegative,

$$|\hat{A}_{\mathscr{J}}| \leqq \operatorname{sp}\hat{A}_{\mathscr{J}} = m \operatorname{Im}\lambda,$$

from which follows (6.1).

To prove (6.2) we first consider the case in which the vector ϕ, having order $r(\leqq \nu_\lambda)$, is orthogonal to the subspace $\mathfrak{Z}((A - \lambda I)^{r-1})$. Then

$$(\phi, \psi) = (\phi, A\psi) = 0,$$

where $\psi = (A - \lambda I)\phi$. Thus

$$(A_{\mathscr{J}}\phi, \psi) = \frac{1}{2i}[(A\phi, \psi) - (\phi, A\psi)] = \frac{1}{2i}|\psi|^2 \tag{6.3}$$

and

$$(A_{\mathscr{J}}\phi, \phi) = \operatorname{Im}(A\phi, \phi) = \operatorname{Im}\lambda|\phi|^2 + (\psi, \phi) = \operatorname{Im}\lambda|\phi|^2. \tag{6.4}$$

According to the Cauchy-Bunjakovskiĭ inequality,

$$|(A_{\mathscr{I}}\phi,\psi)|^2 \leqq |(A_{\mathscr{I}}\phi,\phi)| \cdot |(A_{\mathscr{I}}\psi,\psi)|; \tag{6.5}$$

consequently, by (6.3) and (6.4),

$$\tfrac{1}{4}|\psi|^4 \leqq \operatorname{Im}\lambda|\phi|^2\operatorname{Im}(A\psi,\psi). \tag{6.6}$$

Since $m_\psi = r - 1$, according to the bound (6.1) already proved we have

$$\operatorname{Im}(A\psi,\psi) \leqq (r-1)\operatorname{Im}\lambda|\psi|^2. \tag{6.7}$$

Comparing (6.6) and (6.7), we obtain

$$|\psi| \leqq 2\sqrt{r-1}\operatorname{Im}\lambda|\phi|,$$

i.e.

$$|(A-\lambda I)\phi| \leqq 2\sqrt{r-1}\operatorname{Im}\lambda|\phi|. \tag{6.8}$$

Any root vector ϕ of order $m(\phi \in \mathfrak{Z}((A-\lambda I)^m))$ can be represented in the form

$$\phi = \sum_{j=1}^m \phi_j,$$

where

$$\phi_j \in \mathfrak{Z}((A-\lambda I)^j) \ominus \mathfrak{Z}((A-\lambda I)^{j-1}) \qquad (j = 1,2,\cdots,m),$$

and consequently by virtue of (6.8)

$$\begin{aligned}|(A-\lambda I)\phi| &\leqq \sum_{j=1}^m |(A-\lambda I)\phi_j| \leqq 2\operatorname{Im}\lambda\sum_{j=1}^m\sqrt{j-1}|\phi_j| \\ &\leqq 2\operatorname{Im}\lambda\left(\sum_{j=1}^m(j-1)\right)^{1/2}\sqrt{\sum_{j=1}^m|\phi_j|^2} = \sqrt{2}\operatorname{Im}\lambda\sqrt{m(m-1)}|\phi|.\end{aligned}$$

The lemma is proved.

LEMMA 6.2. *Let ϕ and ψ be root vectors of the dissipative operator A corresponding to the eigenvalues λ_1 and λ_2, respectively. Then*

$$|(\phi,\psi)| \leqq \frac{2\sqrt{nm\operatorname{Im}\lambda_1\operatorname{Im}\lambda_2}}{|\lambda_1-\bar{\lambda}_2|}\sum_{j=0}^{n-1}a_j\sum_{k=0}^{m-1}b^k|\phi|\,|\psi|, \tag{6.9}$$

where $n = m_\phi, m = m_\psi$,

$$a = a_n = \frac{2\sqrt{2}(n-1)\operatorname{Im}\lambda_1}{|\lambda_1-\bar{\lambda}_2|} \quad \textit{and} \quad b = b_m = \frac{2\sqrt{2}(m-1)\operatorname{Im}\lambda_2}{|\lambda_1-\bar{\lambda}_2|}.$$

PROOF. It is easily verified that

$$(A_{\mathscr{I}}\phi,\psi) = \frac{1}{2i}[(\lambda_1-\bar{\lambda}_2)(\phi,\psi) + (\phi_1,\psi) - (\phi,\psi_1)],$$

where

$$\phi_1 = (A - \lambda_1 I)\phi \quad \text{and} \quad \psi_1 = (A - \lambda_2 I)\psi.$$

Hence by the Cauchy-Bunjakovskiĭ inequality (6.5) we have

$$|(\lambda_1 - \bar{\lambda}_2)(\phi, \psi) + (\phi_1, \psi) - (\phi, \psi_1)|^2 \leqq 4\,\mathrm{Im}\,(A\phi, \phi)\,\mathrm{Im}\,(A\psi, \psi).$$

Thus according to (6.1) we have

$$(6.10)\quad |(\lambda_1 - \bar{\lambda}_2)(\phi, \psi)| \leqq 2\sqrt{mn\,\mathrm{Im}\,\lambda_1\,\mathrm{Im}\,\lambda_2}\,|\phi|\,|\psi| + |(\phi_1, \psi)| + |(\phi, \psi_1)|.$$

We shall carry out the rest of the proof of (6.9) by induction on m_ϕ and m_ψ.

We shall first show that if the relation (6.9) holds for all root vectors $\phi \in \mathfrak{L}_{\lambda_1}$ and $\psi \in \mathfrak{L}_{\lambda_2}$ for which $m_\phi = n - 1$, $m_\psi = m$ or $m_\phi = n$, $m_\psi = m - 1$, then it also holds for $m_\phi = n$ and $m_\psi = m$.

In fact, let $m_\phi = n$ and $m_\psi = m$: then $m_{\phi_1} = n - 1$ and $m_{\psi_1} = m - 1$, and so, by the induction hypothesis,

$$|(\phi_1, \psi)| \leqq \frac{2\sqrt{(n-1)\,m\,\mathrm{Im}\,\lambda_1\,\mathrm{Im}\,\lambda_2}}{|\lambda_1 - \bar{\lambda}_2|}\sum_{j=0}^{n-2} a_{n-1}^j \sum_{k=0}^{m-1} b_m^k |\phi_1| \cdot |\psi|.$$

By virtue of (6.2),

$$|\phi_1| \leqq \sqrt{2}\,\mathrm{Im}\,\lambda_1 \sqrt{n(n-1)}\,|\phi|;$$

consequently

$$(6.11)\quad \begin{aligned} |(\phi_1, \psi)| \leqq{}& \frac{2\sqrt{2}\sqrt{nm\,\mathrm{Im}\,\lambda_1\,\mathrm{Im}\,\lambda_2}}{|\lambda_1 - \bar{\lambda}_2|} \\ & \times (n-1)\,\mathrm{Im}\,\lambda_1 \sum_{j=0}^{n-2} a_{n-1}^j \sum_{k=0}^{m-1} b_m^k |\phi|\,|\psi|. \end{aligned}$$

Since

$$2\sqrt{2}(n-1)\,\mathrm{Im}\,\lambda_1/|\lambda_1 - \bar{\lambda}_2| = a_n \quad \text{and} \quad a_{n-1} \leqq a_n,$$

it follows from (6.11) that

$$(6.12)\quad |(\phi_1, \psi)| \leqq \sqrt{nm\,\mathrm{Im}\,\lambda_1\,\mathrm{Im}\,\lambda_2}\sum_{j=1}^{n-1} a_n^j \sum_{k=0}^{m-1} b_m^k |\phi|\,|\psi|.$$

In the same way we derive the relation

$$(6.13)\quad |(\phi, \psi_1)| \leqq \sqrt{nm\,\mathrm{Im}\,\lambda_1\,\mathrm{Im}\,\lambda_2}\sum_{j=0}^{n-1} a_n^j \sum_{k=1}^{m-1} b_m^k |\phi|\,|\psi|.$$

Comparing (6.10), (6.12) and (6.13), we obtain

$$|(\phi, \psi)| \leqq \frac{2\sqrt{nm}\,\mathrm{Im}\,\lambda_1 \mathrm{Im}\,\lambda_2}{|\lambda_1 - \bar{\lambda}_2|}$$

$$\times \left(1 + \frac{1}{2}\sum_{j=1}^{n-1} a_n^j \sum_{k=0}^{m-1} b_m^k + \frac{1}{2}\sum_{j=0}^{n-1} a_n^j \sum_{k=1}^{m-1} b_m^k\right) |\phi|\,|\psi|,$$

from which the bound (6.9) follows in an evident way.

In an entirely analogous way one can prove that if (6.9) holds for the case $m_\phi = n-1$, $m_\psi = 1$, then it holds for $m_\phi = n$, $m_\psi = 1$. In this case the proof simplifies, since $\psi_1 = 0$.

Finally, for $m_\phi = m_\psi = 1$ the vectors ϕ and ψ are eigenvectors of the operator A corresponding to the eigenvalues λ_1 and λ_2 respectively, i.e. $\phi_1 = \psi_1 = 0$. Therefore in this case the inequality (6.9) follows from (6.10). We mention, incidentally, that for the last case the inequality (6.9) was also established in the proof of Theorem 4.1.

The lemma is proved.

As an immediate consequence of Lemma 6.2 we obtain the following result.

THEOREM 6.1 (A. S. MARKUS [2]). *Let λ_1 and λ_2 be two eigenvalues of finite order of the dissipative operator A. Then for the minimal angle between the corresponding root subspaces $\mathfrak{L}_{\lambda_1}$ and $\mathfrak{L}_{\lambda_2}$ one has the bound*

$$\cos\phi(\mathfrak{L}_{\lambda_1}, \mathfrak{L}_{\lambda_2}) \leqq \frac{2\sqrt{m_{\lambda_1} m_{\lambda_2} \mathrm{Im}\,\lambda_1 \mathrm{Im}\,\lambda_2}}{|\lambda_1 - \bar{\lambda}_2|} \sum_{j=0}^{m_{\lambda_1}-1} a_1^j \sum_{k=0}^{m_{\lambda_2}-1} a_2^k,$$

where

$$a_k = 2\sqrt{2}(m_{\lambda k} - 1)\,\mathrm{Im}\,\lambda_k / |\lambda_1 - \bar{\lambda}_2| \qquad (k = 1, 2).$$

2. The basic result of this section is the following theorem.

THEOREM 6.2 (A. S. MARKUS [2]). *Let A be a bounded dissipative operator, with $A_{\mathscr{J}} \in \mathfrak{S}_\infty$, which has a sequence of eigenvalues $\{\lambda_j\}_1^\infty$ $(\lambda_k \neq \lambda_j,\ k \neq j)$. If*

$$(6.14) \qquad \sum_{j,k=1;\, j\neq k}^{\infty} m_j m_k \frac{\mathrm{Im}\,\lambda_j \mathrm{Im}\,\lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < \infty \qquad (m_j = m_{\lambda_j})$$

and

$$(6.15) \qquad \varlimsup_{\substack{j,k\to\infty \\ j\neq k}} (m_j - 1)\frac{\mathrm{Im}\,\lambda_j}{|\lambda_j - \bar{\lambda}_k|} < \frac{1}{2\sqrt{2}},$$

then the sequence of corresponding root subspaces $\{\mathfrak{L}_{\lambda k}\}_1^\infty$ forms a basis of its closed linear hull $\mathfrak{L}$ which is quadratically close to an orthogonal basis, and any sequence $\{\phi_j\}_1^\infty$, obtained as the union of orthonormal bases of the

subspaces $\mathfrak{L}_{\lambda_k}$ $(k = 1, 2, \cdots)$, *forms a basis of the subspace* $\mathfrak{E}$ *equivalent to an orthonormal basis. If the condition* (6.15) *and the strengthened condition*

$$(6.16) \qquad \sum_{j,k=1; j\neq k}^{\infty} \min(\nu_{\lambda_j}, \nu_{\lambda_k})\, m_j m_k \frac{Im\, \lambda_j \operatorname{Im} \lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < \infty \text{ [14]}$$

hold, then the sequence $\{\phi_k\}_1^\infty$ *is a basis of the subspace* $\mathfrak{E}$ *which is quadratically close to an orthonormal basis.*[15]

PROOF. Just as in the proof of Theorem 4.1, we may suppose without loss of generality that the numbers λ_j $(j = 1, 2, \cdots)$ are nonreal. In this case all of the subspaces $\mathfrak{L}_{\lambda_j}$ are finite-dimensional.

It follows from condition (6.15) that the set of numbers

$$M_{jk} = \sum_{r=1}^{m_j - 1} a_{jk}^r \qquad (j, k = 1, 2, \cdots; j \neq k),$$

where

$$a_{jk} = \frac{2\sqrt{2}\,(m_j - 1) \operatorname{Im} \lambda_j}{|\lambda_j - \bar{\lambda}_k|} \qquad (j, k = 1, 2, \cdots; j \neq k),$$

is bounded. Consequently by Theorem 6.1

$$(6.17) \qquad \cos\phi(\mathfrak{L}_{\lambda_j}, \mathfrak{L}_{\lambda_k}) \leq M^2\, \frac{2\sqrt{m_k m_j \operatorname{Im} \lambda_j \operatorname{Im} \lambda_k}}{|\lambda_j - \bar{\lambda}_k|} \quad (j, k = 1, 2, \cdots; j \neq k),$$

where $M = \sup_{j,k} M_{jk}$.

Thus if condition (6.14) is fulfilled, then

$$\sum_{j,k=1; j \neq k}^{\infty} \cos^2\phi(\mathfrak{L}_{\lambda_j}, \mathfrak{L}_{\lambda_k}) < \infty,$$

and so in this case, by Theorem 5.3 and results 1 of §4 and 5 of §5, the sequence of subspaces $\{\mathfrak{L}_{\lambda_j}\}$ forms a basis of $\mathfrak{E}$ quadratically close to an orthogonal one, and any sequence $\{\phi_j\}_1^\infty$ which is the union of orthonormal bases of all of the subspaces $\mathfrak{L}_{\lambda_j}$ $(j = 1, 2, \cdots)$ forms a basis of $\mathfrak{E}$ equivalent to an orthonormal one.

If condition (6.16) is fulfilled, then by (6.17)

$$\sum_{j,k=1; j \neq k}^{\infty} \min(\nu_{\lambda_j}, \nu_{\lambda_k}) \cos^2\phi(\mathfrak{L}_{\lambda_j}, \mathfrak{L}_{\lambda_k}) < \infty,$$

[14] See the footnote to Theorem 4.1.

[15] If $\sup_j m_j < \infty$, the condition (6.15) is superfluous.

and consequently, according to results 6 of §5 and 1 of §4, the sequence $\{\phi_j\}_1^\infty$ forms a basis of the subspace $\mathfrak{E}$ which is quadratically close to an orthonormal one. The theorem is proved.

REMARK 6.1 The theorem remains valid if in its hypotheses (6.14) and (6.16) the lower limit of the summations, which are equal to unity, are replaced by any finite limits.

REMARK 6.2. From the point of view of generalizing well-known results of linear algebra, it would be more natural to choose in every root subspace $\mathfrak{L}_{\lambda_j}$ a basis of vectors forming Jordan chains, and to clarify conditions under which the union of these bases forms a basis of the subspace $\mathfrak{E}$. However, as simple examples show, already in the case $m_{\lambda_k} \geqq 3$ $(k = 1, 2, \cdots)$ no conditions imposed only upon the eigenvalues λ_k $(\operatorname{Im} \lambda_k > 0)$ of the dissipative operator A and on the corresponding numbers $m_{\lambda k}$, $\nu_{\lambda k}$ $(k = 1, 2, \cdots)$ are sufficient for the indicated union of chains to form a basis of the subspace $\mathfrak{E}$.

REMARK 6.3 It is not difficult to see that the proof of Theorem 6.2 remains valid for a closed operator A of the form $A = G + iH$, where G is a selfadjoint operator, and H is a nonnegative completely continuous operator. For this case it is sufficient, in order that the conditions (6.14) and (6.15) be fulfilled, that $H \in \mathfrak{S}_1$ and

$$\inf_{j,k;\, j \neq k} |\lambda_j - \bar{\lambda}_k| = \delta\,(>0).$$

In fact, since

$$\sum_{j=1}^{\infty} m_{\lambda_j} \operatorname{Im} \lambda_j \leqq \sum_{j=1}^{\infty} \nu_{\lambda_j} \operatorname{Im} \lambda_j \leqq \operatorname{sp} H,$$

it follows that

$$\sum_{j,k=1;\, j \neq k}^{\infty} m_{\lambda_j} m_{\lambda k} \frac{\operatorname{Im} \lambda_j \operatorname{Im} \lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} \leqq \delta^{-2} (\operatorname{sp} H)^2$$

and

$$\lim_{\substack{j,k \to \infty \\ j \neq k}} \frac{m_{\lambda_j} \operatorname{Im} \lambda_j}{|\lambda_j - \bar{\lambda}_k|} = 0.$$

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